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5 Trigonometric Functions: Unit Circle Approach

In this chapter and the next we introduce two different but equivalent ways of viewing the trigonometric functions. One way is to view them as *functions of real numbers* (Chapter 5); the other is to view them as *functions of angles* (Chapter 6). The two approaches to trigonometry are independent of each other, so either Chapter 5 or Chapter 6 may be studied first. The applications of trigonometry are numerous, including signal processing, digital coding of music and videos, finding distances to stars, producing CAT scans for medical imaging, and many others. These applications are very diverse, and we need to study both approaches to trigonometry because the different approaches are required for different applications.

One of the main applications of trigonometry that we study in this chapter is periodic motion. If you've ever taken a Ferris wheel ride, then you know about periodic motion—that is, motion that repeats over and over. Periodic motion occurs often in nature, as in the daily rising and setting of the sun, the daily variation in tide levels, the vibrations of a leaf in the wind, and many more. We will see in this chapter how the trigonometric functions are used to model periodic motion.

5.1 THE UNIT CIRCLE

■ The Unit Circle ■ Terminal Points on the Unit Circle ■ The Reference Number

In this section we explore some properties of the circle of radius 1 centered at the origin. These properties are used in the next section to define the trigonometric functions.

■ The Unit Circle

The set of points at a distance 1 from the origin is a circle of radius 1 (see Figure 1). In Section 1.9 we learned that the equation of this circle is $x^2 + y^2 = 1$.

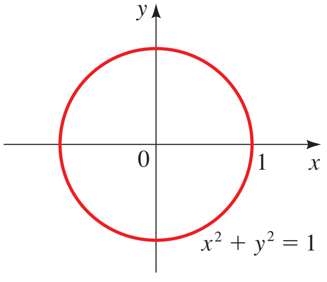


FIGURE 1 The unit circle

THE UNIT CIRCLE

The **unit circle** is the circle of radius 1 centered at the origin in the *xy*-plane. Its equation is

$$x^2 + y^2 = 1$$

Circles are studied in Section 1.9, page 97.

EXAMPLE 1 ■ A Point on the Unit Circle

Show that the point $P\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)$ is on the unit circle.

SOLUTION We need to show that this point satisfies the equation of the unit circle, that is, $x^2 + y^2 = 1$. Since

$$\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{3}{9} + \frac{6}{9} = 1$$

P is on the unit circle.

Now Try Exercise 3 ■

EXAMPLE 2 ■ Locating a Point on the Unit Circle

The point $P(\sqrt{3}/2, y)$ is on the unit circle in Quadrant IV. Find its *y*-coordinate.

SOLUTION Since the point is on the unit circle, we have

$$\left(\frac{\sqrt{3}}{2}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

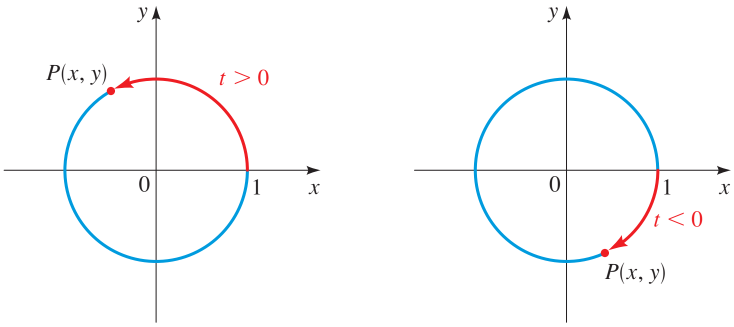
$$y = \pm\frac{1}{2}$$

Since the point is in Quadrant IV, its *y*-coordinate must be negative, so $y = -\frac{1}{2}$.

Now Try Exercise 9 ■

■ Terminal Points on the Unit Circle

Suppose t is a real number. If $t \geq 0$, let's mark off a distance t along the unit circle, starting at the point $(1, 0)$ and moving in a counterclockwise direction. If $t < 0$, we mark off a distance $|t|$ in a clockwise direction (Figure 2). In this way we arrive at a point $P(x, y)$ on the unit circle. The point $P(x, y)$ obtained in this way is called the **terminal point** determined by the real number t.



(a) Terminal point $P(x, y)$ determined by $t > 0$

(b) Terminal point $P(x, y)$ determined by $t < 0$

FIGURE 2

The circumference of the unit circle is $C = 2\pi(1) = 2\pi$. So if a point starts at $(1, 0)$ and moves counterclockwise all the way around the unit circle and returns to $(1, 0)$, it travels a distance of 2π. To move halfway around the circle, it travels a distance of $\frac{1}{2}(2\pi) = \pi$. To move a quarter of the distance around the circle, it travels a distance of $\frac{1}{4}(2\pi) = \pi/2$. Where does the point end up when it travels these distances along the circle? From Figure 3 we see, for example, that when it travels a distance of π starting at $(1, 0)$, its terminal point is $(-1, 0)$.

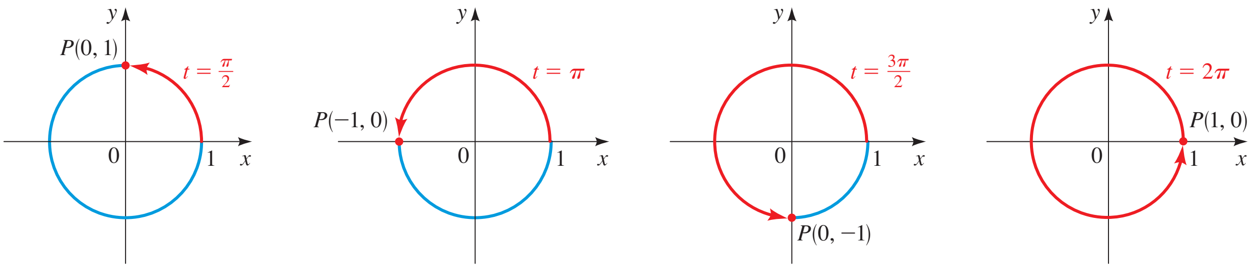


FIGURE 3 Terminal points determined by $t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π

EXAMPLE 3 ■ Finding Terminal Points

Find the terminal point on the unit circle determined by each real number t.

(a) $t = 3\pi$ **(b)** $t = -\pi$ **(c)** $t = -\dfrac{\pi}{2}$

SOLUTION From Figure 4 we get the following:

(a) The terminal point determined by 3π is $(-1, 0)$.

(b) The terminal point determined by $-\pi$ is $(-1, 0)$.

(c) The terminal point determined by $-\pi/2$ is $(0, -1)$.

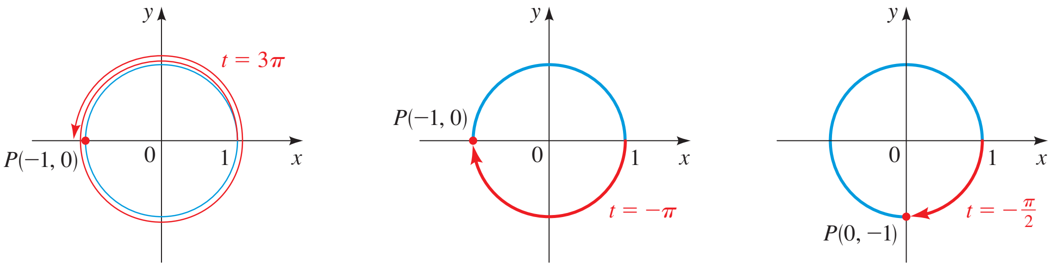


FIGURE 4

Notice that different values of t can determine the same terminal point.

Now Try Exercise 23 ■

The terminal point $P(x, y)$ determined by $t = \pi/4$ is the same distance from $(1, 0)$ as from $(0, 1)$ along the unit circle (see Figure 5).

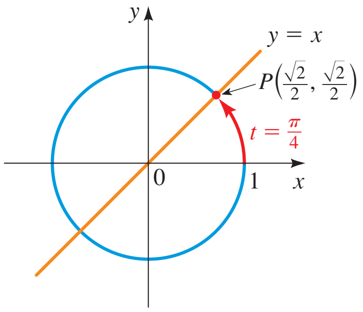


FIGURE 5

Since the unit circle is symmetric with respect to the line $y = x$, it follows that P lies on the line $y = x$. So P is the point of intersection (in the Quadrant I) of the circle $x^2 + y^2 = 1$ and the line $y = x$. Substituting x for y in the equation of the circle, we get

$$x^2 + x^2 = 1$$

$$2x^2 = 1 \qquad \text{Combine like terms}$$

$$x^2 = \frac{1}{2} \qquad \text{Divide by 2}$$

$$x = \pm\frac{1}{\sqrt{2}} \qquad \text{Take square roots}$$

Since P is in the Quadrant I, $x = 1/\sqrt{2}$ and since $y = x$, we have $y = 1/\sqrt{2}$ also. Thus the terminal point determined by $\pi/4$ is

$$P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Similar methods can be used to find the terminal points determined by $t = \pi/6$ and $t = \pi/3$ (see Exercises 61 and 62). Table 1 and Figure 6 give the terminal points for some special values of t.

TABLE 1

t	Terminal point determined by t
0	$(1, 0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0, 1)$

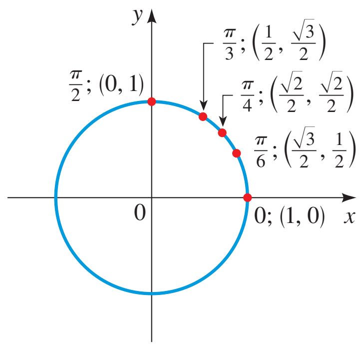


FIGURE 6

EXAMPLE 4 ■ Finding Terminal Points

Find the terminal point determined by each given real number t.

(a) $t = -\frac{\pi}{4}$ **(b)** $t = \frac{3\pi}{4}$ **(c)** $t = -\frac{5\pi}{6}$

SOLUTION

(a) Let P be the terminal point determined by $-\pi/4$, and let Q be the terminal point determined by $\pi/4$. From Figure 7(a) we see that the point P has the same coordinates as Q except for sign. Since P is in Quadrant IV, its x-coordinate is positive and its y-coordinate is negative. Thus, the terminal point is $P\left(\sqrt{2}/2, -\sqrt{2}/2\right)$.

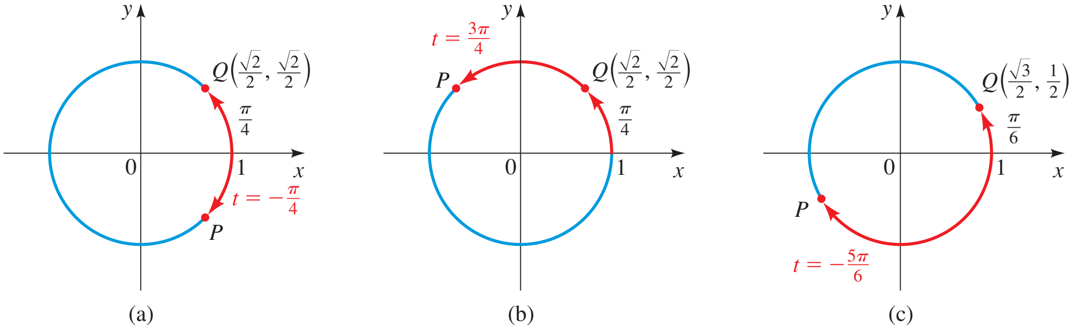


FIGURE 7

(b) Let P be the terminal point determined by $3\pi/4$, and let Q be the terminal point determined by $\pi/4$. From Figure 7(b) we see that the point P has the same coordinates as Q except for sign. Since P is in Quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus the terminal point is $P\left(-\sqrt{2}/2, \sqrt{2}/2\right)$.

(c) Let P be the terminal point determined by $-5\pi/6$, and let Q be the terminal point determined by $\pi/6$. From Figure 7(c) we see that the point P has the same coordinates as Q except for sign. Since P is in Quadrant III, its coordinates are both negative. Thus the terminal point is $P\left(-\sqrt{3}/2, -\frac{1}{2}\right)$.

Now Try Exercise 27 ■

The Reference Number

From Examples 3 and 4 we see that to find a terminal point in any quadrant we need only know the "corresponding" terminal point in the first quadrant. We use the idea of the *reference number* to help us find terminal points.

REFERENCE NUMBER

Let t be a real number. The **reference number** $\bar{t}$ associated with t is the shortest distance along the unit circle between the terminal point determined by t and the x-axis.

Figure 8 shows that to find the reference number $\bar{t}$, it's helpful to know the quadrant in which the terminal point determined by t lies. If the terminal point lies in Quadrant I or IV, where x is positive, we find $\bar{t}$ by moving along the circle to the *positive* x-axis. If it lies in Quadrant II or III, where x is negative, we find $\bar{t}$ by moving along the circle to the *negative* x-axis.

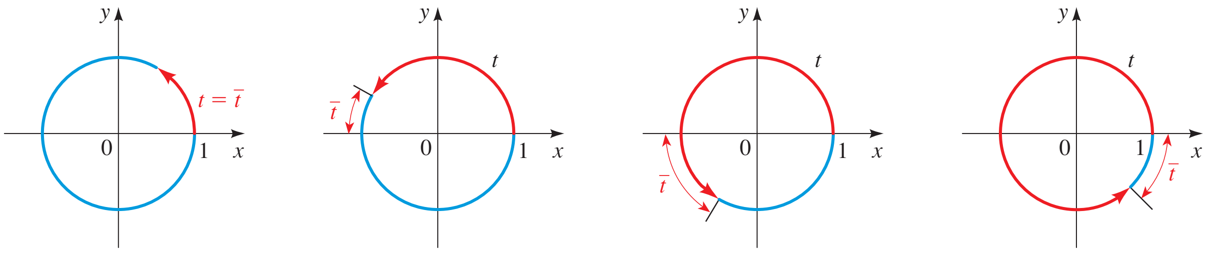


FIGURE 8 The reference number $\bar{t}$ for t

EXAMPLE 5 ■ Finding Reference Numbers

Find the reference number for each value of t.

(a) $t = \frac{5\pi}{6}$ **(b)** $t = \frac{7\pi}{4}$ **(c)** $t = -\frac{2\pi}{3}$ **(d)** $t = 5.80$

SOLUTION From Figure 9 we find the reference numbers as follows.

(a) $\bar{t} = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$ **(b)** $\bar{t} = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$

(c) $\bar{t} = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$ **(d)** $\bar{t} = 2\pi - 5.80 \approx 0.48$

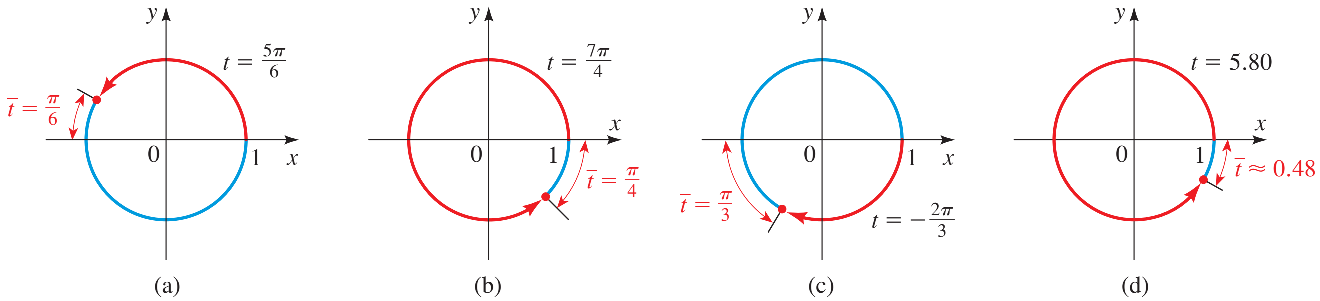


FIGURE 9

Now Try Exercise 37 ■

USING REFERENCE NUMBERS TO FIND TERMINAL POINTS

To find the terminal point P determined by any value of t, we use the following steps:

1. Find the reference number $\bar{t}$.
2. Find the terminal point $Q(a, b)$ determined by $\bar{t}$.
3. The terminal point determined by t is $P(\pm a, \pm b)$, where the signs are chosen according to the quadrant in which this terminal point lies.

EXAMPLE 6 ■ Using Reference Numbers to Find Terminal Points

Find the terminal point determined by each given real number t.

(a) $t = \frac{5\pi}{6}$ **(b)** $t = \frac{7\pi}{4}$ **(c)** $t = -\frac{2\pi}{3}$

SOLUTION The reference numbers associated with these values of t were found in Example 5.

(a) The reference number is $\bar{t} = \pi/6$, which determines the terminal point $\left(\sqrt{3}/2, \frac{1}{2}\right)$ from Table 1. Since the terminal point determined by t is in Quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus the desired terminal point is

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

(b) The reference number is $\bar{t} = \pi/4$, which determines the terminal point $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ from Table 1. Since the terminal point is in Quadrant IV, its x-coordinate is positive and its y-coordinate is negative. Thus the desired terminal point is

$$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

(c) The reference number is $\bar{t} = \pi/3$, which determines the terminal point $\left(\frac{1}{2}, \sqrt{3}/2\right)$ from Table 1. Since the terminal point determined by t is in Quadrant III, its coordinates are both negative. Thus the desired terminal point is

$$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Now Try Exercise 41 ■

Since the circumference of the unit circle is 2π, the terminal point determined by t is the same as that determined by $t + 2\pi$ or $t - 2\pi$. In general, we can add or subtract 2π any number of times without changing the terminal point determined by t. We use this observation in the next example to find terminal points for large t.

EXAMPLE 7 ■ Finding the Terminal Point for Large *t*

Find the terminal point determined by $t = \dfrac{29\pi}{6}$.

SOLUTION Since

$$t = \frac{29\pi}{6} = 4\pi + \frac{5\pi}{6}$$

we see that the terminal point of t is the same as that of $5\pi/6$ (that is, we subtract 4π). So by Example 6(a) the terminal point is $\left(-\sqrt{3}/2, \frac{1}{2}\right)$. (See Figure 10.)

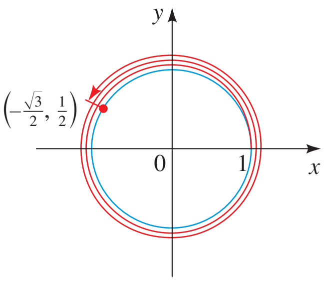


FIGURE 10

Now Try Exercise 47 ■

5.1 EXERCISES

CONCEPTS

1. **(a)** The unit circle is the circle centered at ________ with radius ________.
   (b) The equation of the unit circle is ____________.
   (c) Suppose the point $P(x, y)$ is on the unit circle. Find the missing coordinate:
   (i) $P(1, \square)$ (ii) $P(\square, 1)$
   (iii) $P(-1, \square)$ (iv) $P(\square, -1)$
2. **(a)** If we mark off a distance t along the unit circle, starting at $(1, 0)$ and moving in a counterclockwise direction, we arrive at the ____________ point determined by t.
   (b) The terminal points determined by $\pi/2$, π, $-\pi/2$, 2π are ________, ________, ________, and ________, respectively.

SKILLS

3–8 ■ Points on the Unit Circle Show that the point is on the unit circle.

3. $\left(\frac{3}{5}, -\frac{4}{5}\right)$
4. $\left(-\frac{24}{25}, -\frac{7}{25}\right)$
5. $\left(\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$
6. $\left(-\frac{5}{7}, -\frac{2\sqrt{6}}{7}\right)$
7. $\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$
8. $\left(\frac{\sqrt{11}}{6}, \frac{5}{6}\right)$

9–14 ■ Points on the Unit Circle Find the missing coordinate of P, using the fact that P lies on the unit circle in the given quadrant.

Coordinates	Quadrant
9. $P\left(-\frac{3}{5}, \square\right)$	III
10. $P\left(\square, -\frac{7}{25}\right)$	IV
11. $P\left(\square, \frac{1}{3}\right)$	II

Coordinates	Quadrant
12. $P\left(\frac{2}{5}, \square\right)$	I
13. $P\left(\square, -\frac{2}{7}\right)$	IV
14. $P\left(-\frac{2}{3}, \square\right)$	II

15–20 ■ Points on the Unit Circle The point P is on the unit circle. Find $P(x, y)$ from the given information.

15. The x-coordinate of P is $\frac{5}{13}$, and the y-coordinate is negative.
16. The y-coordinate of P is $-\frac{3}{5}$, and the x-coordinate is positive.

17. The y-coordinate of P is $\frac{2}{3}$, and the x-coordinate is negative.

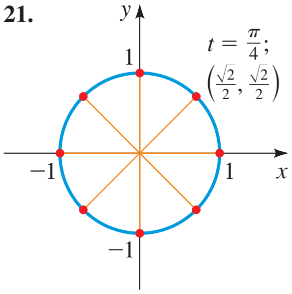
18. The x-coordinate of P is positive, and the y-coordinate of P is $-\sqrt{5}/5$.

19. The x-coordinate of P is $-\sqrt{2}/3$, and P lies below the x-axis.

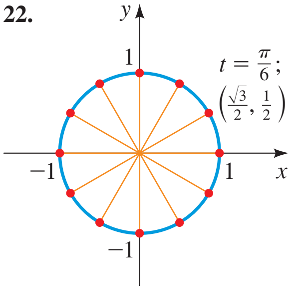
20. The x-coordinate of P is $-\frac{2}{5}$, and P lies above the x-axis.

21–22 ■ Terminal Points Find t and the terminal point determined by t for each point in the figure. In Exercise 21, t increases in increments of $\pi/4$; in Exercise 22, t increases in increments of $\pi/6$.

21.



22.



23–36 ■ Terminal Points Find the terminal point $P(x, y)$ on the unit circle determined by the given value of t.

23. $t = 4\pi$

24. $t = -3\pi$

25. $t = \dfrac{3\pi}{2}$

26. $t = \dfrac{5\pi}{2}$

27. $t = -\dfrac{\pi}{6}$

28. $t = \dfrac{7\pi}{6}$

29. $t = \dfrac{5\pi}{4}$

30. $t = \dfrac{4\pi}{3}$

31. $t = -\dfrac{7\pi}{6}$

32. $t = \dfrac{5\pi}{3}$

33. $t = -\dfrac{7\pi}{4}$

34. $t = -\dfrac{4\pi}{3}$

35. $t = -\dfrac{3\pi}{4}$

36. $t = \dfrac{11\pi}{6}$

37–40 ■ Reference Numbers Find the reference number for each value of t.

37. **(a)** $t = \dfrac{4\pi}{3}$ **(b)** $t = \dfrac{5\pi}{3}$ **(c)** $t = -\dfrac{7\pi}{6}$ **(d)** $t = 3.5$

38. **(a)** $t = 9\pi$ **(b)** $t = -\dfrac{5\pi}{4}$ **(c)** $t = \dfrac{25\pi}{6}$ **(d)** $t = 4$

39. **(a)** $t = \dfrac{5\pi}{7}$ **(b)** $t = -\dfrac{7\pi}{9}$ **(c)** $t = -3$ **(d)** $t = 5$

40. **(a)** $t = \dfrac{11\pi}{5}$ **(b)** $t = -\dfrac{9\pi}{7}$ **(c)** $t = 6$ **(d)** $t = -7$

41–54 ■ Terminal Points and Reference Numbers Find **(a)** the reference number for each value of t and **(b)** the terminal point determined by t.

41. $t = \dfrac{11\pi}{6}$

42. $t = \dfrac{2\pi}{3}$

43. $t = -\dfrac{4\pi}{3}$

44. $t = \dfrac{5\pi}{3}$

45. $t = -\dfrac{2\pi}{3}$

46. $t = -\dfrac{7\pi}{6}$

47. $t = \dfrac{13\pi}{4}$

48. $t = \dfrac{13\pi}{6}$

49. $t = \dfrac{41\pi}{6}$

50. $t = \dfrac{17\pi}{4}$

51. $t = -\dfrac{11\pi}{3}$

52. $t = \dfrac{31\pi}{6}$

53. $t = \dfrac{16\pi}{3}$

54. $t = -\dfrac{41\pi}{4}$

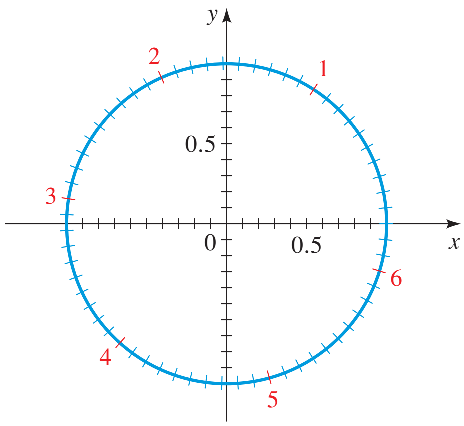
55–58 ■ Terminal Points The unit circle is graphed in the figure below. Use the figure to find the terminal point determined by the real number t, with coordinates rounded to one decimal place.

55. $t = 1$

56. $t = 2.5$

57. $t = -1.1$

58. $t = 4.2$



SKILLS Plus

59. Terminal Points Suppose that the terminal point determined by t is the point $\left(\frac{3}{5}, \frac{4}{5}\right)$ on the unit circle. Find the terminal point determined by each of the following.

(a) $\pi - t$ **(b)** $-t$

(c) $\pi + t$ **(d)** $2\pi + t$

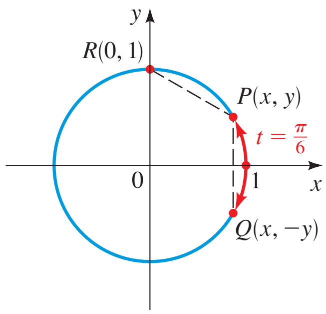
60. Terminal Points Suppose that the terminal point determined by t is the point $\left(\frac{3}{4}, \sqrt{7}/4\right)$ on the unit circle. Find the terminal point determined by each of the following.

(a) $-t$ **(b)** $4\pi + t$

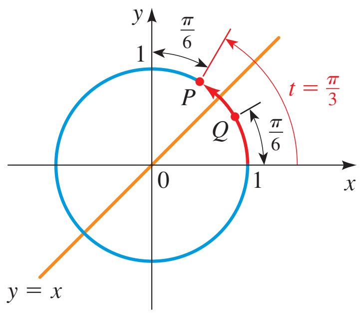
(c) $\pi - t$ **(d)** $t - \pi$

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

61. DISCOVER ■ PROVE: Finding the Terminal Point for $\pi/6$ Suppose the terminal point determined by $t = \pi/6$ is $P(x, y)$ and the points Q and R are as shown in the figure. Why are the distances PQ and PR the same? Use this fact, together with the Distance Formula, to show that the coordinates of P satisfy the equation $2y = \sqrt{x^2 + (y - 1)^2}$. Simplify this equation using the fact that $x^2 + y^2 = 1$. Solve the simplified equation to find $P(x, y)$.



62. DISCOVER ■ PROVE: Finding the Terminal Point for $\pi/3$ Now that you know the terminal point determined by $t = \pi/6$, use symmetry to find the terminal point determined by $t = \pi/3$ (see the figure). Explain your reasoning.



5.2 TRIGONOMETRIC FUNCTIONS OF REAL NUMBERS

■ The Trigonometric Functions ■ Values of the Trigonometric Functions
■ Fundamental Identities

A function is a rule that assigns to each real number another real number. In this section we use properties of the unit circle from the preceding section to define the trigonometric functions.

■ The Trigonometric Functions

Recall that to find the terminal point $P(x, y)$ for a given real number t, we move a distance $|t|$ along the unit circle, starting at the point $(1, 0)$. We move in a counterclockwise direction if t is positive and in a clockwise direction if t is negative (see Figure 1). We now use the x- and y-coordinates of the point $P(x, y)$ to define several functions. For instance, we define the function called *sine* by assigning to each real number t the y-coordinate of the terminal point $P(x, y)$ determined by t. The functions *cosine*, *tangent*, *cosecant*, *secant*, and *cotangent* are also defined by using the coordinates of $P(x, y)$.

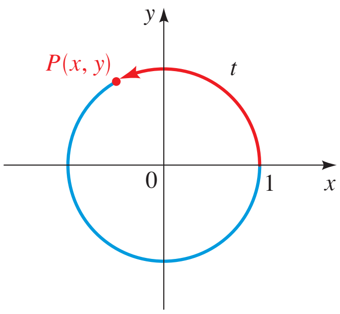


FIGURE 1

DEFINITION OF THE TRIGONOMETRIC FUNCTIONS

Let t be any real number and let $P(x, y)$ be the terminal point on the unit circle determined by t. We define

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x} \quad (x \neq 0)$$

$$\csc t = \frac{1}{y} \quad (y \neq 0) \qquad \sec t = \frac{1}{x} \quad (x \neq 0) \qquad \cot t = \frac{x}{y} \quad (y \neq 0)$$

Because the trigonometric functions can be defined in terms of the unit circle, they are sometimes called the **circular functions**.

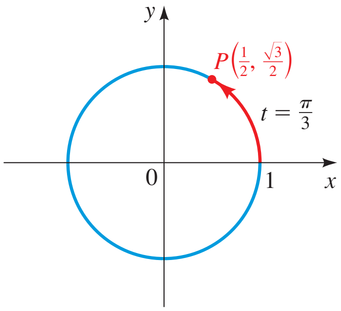


FIGURE 2

EXAMPLE 1 ■ Evaluating Trigonometric Functions

Find the six trigonometric functions of each given real number t.

(a) $t = \dfrac{\pi}{3}$ **(b)** $t = \dfrac{\pi}{2}$

SOLUTION

(a) From Table 1 on page 404, we see that the terminal point determined by $t = \pi/3$ is $P(\frac{1}{2}, \sqrt{3}/2)$. (See Figure 2.) Since the coordinates are $x = \frac{1}{2}$ and $y = \sqrt{3}/2$, we have

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2} \qquad \tan\frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \qquad \sec\frac{\pi}{3} = 2 \qquad \cot\frac{\pi}{3} = \frac{1/2}{\sqrt{3}/2} = \frac{\sqrt{3}}{3}$$

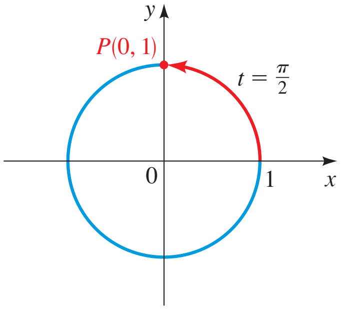


FIGURE 3

(b) The terminal point determined by $\pi/2$ is $P(0, 1)$. (See Figure 3.) So

$$\sin\frac{\pi}{2} = 1 \qquad \cos\frac{\pi}{2} = 0 \qquad \csc\frac{\pi}{2} = \frac{1}{1} = 1 \qquad \cot\frac{\pi}{2} = \frac{0}{1} = 0$$

But $\tan \pi/2$ and $\sec \pi/2$ are undefined because $x = 0$ appears in the denominator in each of their definitions.

Now Try Exercise 3 ■

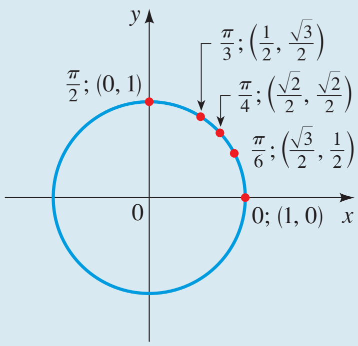
Some special values of the trigonometric functions are listed in the table below. This table is easily obtained from Table 1 of Section 5.1, together with the definitions of the trigonometric functions.

SPECIAL VALUES OF THE TRIGONOMETRIC FUNCTIONS

The following values of the trigonometric functions are obtained from the special terminal points.

TABLE 1

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0



We can easily remember the sines and cosines of the basic angles by writing them in the form $\sqrt{\blacksquare}/2$:

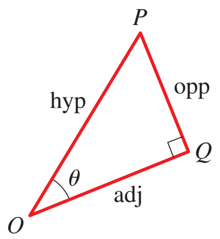
t	$\sin t$	$\cos t$
0	$\sqrt{0}/2$	$\sqrt{4}/2$
$\pi/6$	$\sqrt{1}/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$\sqrt{1}/2$
$\pi/2$	$\sqrt{4}/2$	$\sqrt{0}/2$

Example 1 shows that some of the trigonometric functions fail to be defined for certain real numbers. So we need to determine their domains. The functions sine and cosine are defined for all values of t. Since the functions cotangent and cosecant have y in the denominator of their definitions, they are not defined whenever the y-coordinate of the terminal point $P(x, y)$ determined by t is 0. This happens when $t = n\pi$ for any integer n, so their domains do not include these points. The functions tangent and secant have x in the denominator in their definitions, so they are not defined whenever $x = 0$. This happens when $t = (\pi/2) + n\pi$ for any integer n.

(*text continues on page 412*)

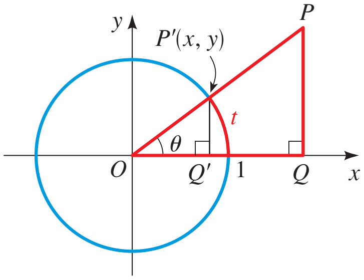
Relationship to the Trigonometric Functions of Angles

If you have studied the trigonometry of right triangles in Chapter 6, you are probably wondering how the sine and cosine of an *angle* relate to those of this section. To see how, let's start with a right triangle, ΔOPQ.



Right triangle *OPQ*

Place the triangle in the coordinate plane as shown, with angle θ in standard position.



$P'(x, y)$ is the terminal point determined by t.

The point $P'(x, y)$ in the figure is the terminal point determined by t. Note that triangle OPQ is similar to the small triangle $OP'Q'$ whose legs have lengths x and y.

Now, by the definition of the trigonometric functions of the *angle* θ we have

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{PQ}{OP} = \frac{P'Q'}{OP'}$$

$$= \frac{y}{1} = y$$

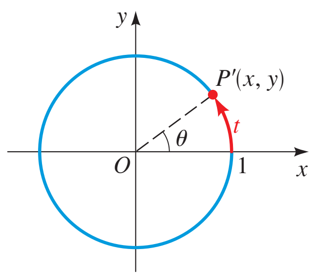
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{OQ}{OP} = \frac{OQ'}{OP'}$$

$$= \frac{x}{1} = x$$

By the definition of the trigonometric functions of the *real number t*, we have

$$\sin t = y \qquad \cos t = x$$

Now, if θ is measured in radians, then $\theta = t$ (see the figure). So the trigonometric functions of the angle with radian measure θ are exactly the same as the trigonometric functions defined in terms of the terminal point determined by the real number t.



The radian measure of angle θ is t.

Why then study trigonometry in two different ways? Because different applications require that we view the trigonometric functions differently. (Compare Section 5.6 with Sections 6.2, 6.5, and 6.6.)

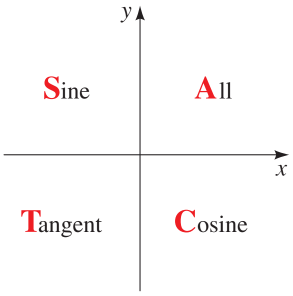
DOMAINS OF THE TRIGONOMETRIC FUNCTIONS

Function	Domain
sin, cos	All real numbers
tan, sec	All real numbers other than $\frac{\pi}{2} + n\pi$ for any integer n
cot, csc	All real numbers other than $n\pi$ for any integer, n

Values of the Trigonometric Functions

To compute values of the trigonometric functions for any real number t, we first determine their signs. The signs of the trigonometric functions depend on the quadrant in which the terminal point of t lies. For example, if the terminal point $P(x, y)$ determined by t lies in Quadrant III, then its coordinates are both negative. So sin t, cos t, csc t, and sec t are all negative, whereas tan t and cot t are positive. You can check the other entries in the following box.

The following mnemonic device will help you remember which trigonometric functions are positive in each quadrant: **A**ll of them, **S**ine, **T**angent, or **C**osine.



You can remember this as "All Students Take Calculus."

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

For example $\cos(2\pi/3) < 0$ because the terminal point of $t = 2\pi/3$ is in Quadrant II, whereas $\tan 4 > 0$ because the terminal point of $t = 4$ is in Quadrant III.

In Section 5.1 we used the reference number to find the terminal point determined by a real number t. Since the trigonometric functions are defined in terms of the coordinates of terminal points, we can use the reference number to find values of the trigonometric functions. Suppose that $\bar{t}$ is the reference number for t. Then the terminal point of $\bar{t}$ has the same coordinates, except possibly for sign, as the terminal point of t. So the value of each trigonometric function at t is the same, except possibly for sign, as its value at $\bar{t}$. We illustrate this procedure in the next example.

EVALUATING TRIGONOMETRIC FUNCTIONS FOR ANY REAL NUMBER

To find the values of the trigonometric functions for any real number t, we carry out the following steps.

1. **Find the reference number**. Find the reference number $\bar{t}$ associated with t.
2. **Find the sign**. Determine the sign of the trigonometric function of t by noting the quadrant in which the terminal point lies.
3. **Find the value**. The value of the trigonometric function of t is the same, except possibly for sign, as the value of the trigonometric function of $\bar{t}$.

EXAMPLE 2 ■ Evaluating Trigonometric Functions

Find each value.

(a) $\cos \dfrac{2\pi}{3}$ **(b)** $\tan\left(-\dfrac{\pi}{3}\right)$ **(c)** $\sin \dfrac{19\pi}{4}$

SOLUTION

(a) The reference number for $2\pi/3$ is $\pi/3$ (see Figure 4(a)). Since the terminal point of $2\pi/3$ is in Quadrant II, $\cos(2\pi/3)$ is negative. Thus

$$\cos \frac{2\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$$

Sign | Reference number | From Table 1 (page 410)

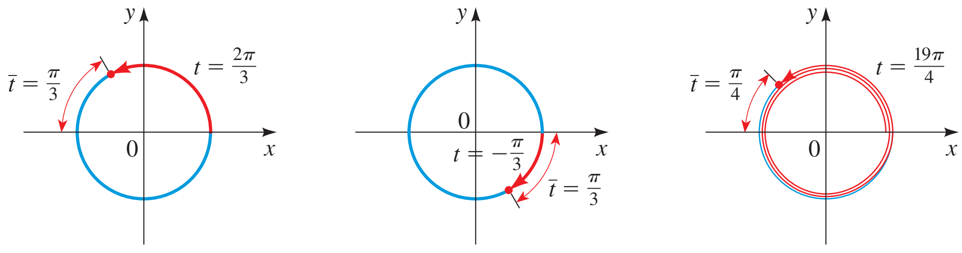


FIGURE 4 (a) (b) (c)

(b) The reference number for $-\pi/3$ is $\pi/3$ (see Figure 4(b)). Since the terminal point of $-\pi/3$ is in Quadrant IV, $\tan(-\pi/3)$ is negative. Thus

$$\tan\left(-\frac{\pi}{3}\right) = -\tan \frac{\pi}{3} = -\sqrt{3}$$

Sign | Reference number | From Table 1 (page 410)

(c) Since $(19\pi/4) - 4\pi = 3\pi/4$, the terminal points determined by $19\pi/4$ and $3\pi/4$ are the same. The reference number for $3\pi/4$ is $\pi/4$ (see Figure 4(c)). Since the terminal point of $3\pi/4$ is in Quadrant II, $\sin(3\pi/4)$ is positive. Thus

$$\sin \frac{19\pi}{4} = \sin \frac{3\pi}{4} = +\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Subtract 4π | Sign | Reference number | From Table 1 (page 410)

Now Try Exercise 5 ■

So far, we have been able to compute the values of the trigonometric functions only for certain values of t. In fact, we can compute the values of the trigonometric functions whenever t is a multiple of $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$. How can we compute the trigonometric functions for other values of t? For example, how can we find $\sin 1.5$? One way is to carefully sketch a diagram and read the value (see Exercises 37–44); however, this method is not very accurate. Fortunately, programmed directly into scientific calculators are mathematical procedures (see the margin note on page 433) that find the values of *sine*, *cosine*, and *tangent* correct to the number of digits in the display. The calculator must be put in *radian mode* to evaluate these functions. To find values of cosecant, secant, and cotangent using a calculator, we need to use the following *reciprocal relations*:

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t}$$

These identities follow from the definitions of the trigonometric functions. For instance, since $\sin t = y$ and $\csc t = 1/y$, we have $\csc t = 1/y = 1/(\sin t)$. The others follow similarly.

EXAMPLE 3 ■ Using a Calculator to Evaluate Trigonometric Functions

Using a calculator, find the following.

(a) $\sin 2.2$ **(b)** $\cos 1.1$ **(c)** $\cot 28$ **(d)** $\csc 0.98$

SOLUTION Making sure our calculator is set to radian mode and rounding the results to six decimal places, we get

(a) $\sin 2.2 \approx 0.808496$ **(b)** $\cos 1.1 \approx 0.453596$

(c) $\cot 28 = \dfrac{1}{\tan 28} \approx -3.553286$ **(d)** $\csc 0.98 = \dfrac{1}{\sin 0.98} \approx 1.204098$

Now Try Exercises 39 and 41 ■

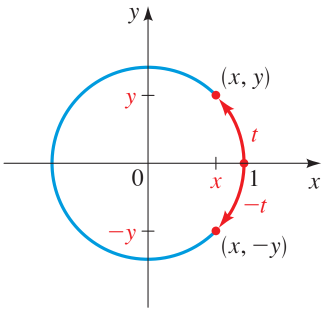


FIGURE 5

Let's consider the relationship between the trigonometric functions of t and those of $-t$. From Figure 5 we see that

$$\sin(-t) = -y = -\sin t$$

$$\cos(-t) = x = \cos t$$

$$\tan(-t) = \frac{-y}{x} = -\frac{y}{x} = -\tan t$$

These equations show that sine and tangent are odd functions, whereas cosine is an even function. It's easy to see that the reciprocal of an even function is even and the reciprocal of an odd function is odd. This fact, together with the reciprocal relations, completes our knowledge of the even-odd properties for all the trigonometric functions.

Even and odd functions are defined in Section 2.6.

EVEN-ODD PROPERTIES

Sine, cosecant, tangent, and cotangent are odd functions; cosine and secant are even functions.

$$\sin(-t) = -\sin t \qquad \cos(-t) = \cos t \qquad \tan(-t) = -\tan t$$

$$\csc(-t) = -\csc t \qquad \sec(-t) = \sec t \qquad \cot(-t) = -\cot t$$

EXAMPLE 4 ■ Even and Odd Trigonometric Functions

Use the even-odd properties of the trigonometric functions to determine each value.

(a) $\sin\left(-\dfrac{\pi}{6}\right)$ **(b)** $\cos\left(-\dfrac{\pi}{4}\right)$

SOLUTION By the even-odd properties and Table 1 on page 410, we have

(a) $\sin\left(-\dfrac{\pi}{6}\right) = -\sin\dfrac{\pi}{6} = -\dfrac{1}{2}$ Sine is odd

(b) $\cos\left(-\dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$ Cosine is even

Now Try Exercise 13 ■

■ Fundamental Identities

The trigonometric functions are related to each other through equations called **trigonometric identities**. We give the most important ones in the following box.*

FUNDAMENTAL IDENTITIES

Reciprocal Identities

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t} \qquad \tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

Pythagorean Identities

$$\sin^2 t + \cos^2 t = 1 \qquad \tan^2 t + 1 = \sec^2 t \qquad 1 + \cot^2 t = \csc^2 t$$

Proof The reciprocal identities follow immediately from the definitions on page 409. We now prove the Pythagorean identities. By definition $\cos t = x$ and $\sin t = y$, where x and y are the coordinates of a point $P(x, y)$ on the unit circle. Since $P(x, y)$ is on the unit circle, we have $x^2 + y^2 = 1$. Thus

$$\sin^2 t + \cos^2 t = 1$$

Dividing both sides by $\cos^2 t$ (provided that $\cos t \neq 0$), we get

$$\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

$$\left(\frac{\sin t}{\cos t}\right)^2 + 1 = \left(\frac{1}{\cos t}\right)^2$$

$$\tan^2 t + 1 = \sec^2 t$$

We have used the reciprocal identities $\sin t/\cos t = \tan t$ and $1/\cos t = \sec t$. Similarly, dividing both sides of the first Pythagorean identity by $\sin^2 t$ (provided that $\sin t \neq 0$) gives us $1 + \cot^2 t = \csc^2 t$. ■

As their name indicates, the fundamental identities play a central role in trigonometry because we can use them to relate any trigonometric function to any other. So if we know the value of any one of the trigonometric functions at t, then we can find the values of all the others at t.

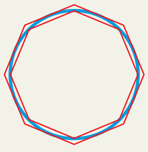
EXAMPLE 5 ■ Finding All Trigonometric Functions from the Value of One

If $\cos t = \frac{3}{5}$ and t is in Quadrant IV, find the values of all the trigonometric functions at t.

*We follow the usual convention of writing $\sin^2 t$ for $(\sin t)^2$. In general, we write $\sin^n t$ for $(\sin t)^n$ for all integers n except $n = -1$. The superscript $n = -1$ will be assigned another meaning in Section 5.5. Of course, the same convention applies to the other five trigonometric functions.

The Value of π

The number π is the ratio of the circumference of a circle to its diameter. It has been known since ancient times that this ratio is the same for all circles. The first systematic effort to find a numerical approximation for π was made by Archimedes (ca. 240 B.C.), who proved that $\frac{22}{7} < \pi < \frac{223}{71}$ by finding the perimeters of regular polygons inscribed in and circumscribed about a circle.



In about A.D. 480, the Chinese physicist Tsu Ch'ung-chih gave the approximation

$$\pi \approx \frac{355}{113} = 3.141592\ldots$$

which is correct to six decimals. This remained the most accurate estimation of π until the Dutch mathematician Adrianus Romanus (1593) used polygons with more than a billion sides to compute π correct to 15 decimals. In the 17th century, mathematicians began to use infinite series and trigonometric identities in the quest for π. The Englishman William Shanks spent 15 years (1858–1873) using these methods to compute π to 707 decimals, but in 1946 it was found that his figures were wrong beginning with the 528th decimal. Today, with the aid of computers, mathematicians routinely determine π correct to millions of decimals. In fact, mathematicians have recently developed new algorithms that can be programmed into computers to calculate π to many trillions of decimal places.

SOLUTION From the Pythagorean identities we have

$$\sin^2 t + \cos^2 t = 1$$
$$\sin^2 t + \left(\tfrac{3}{5}\right)^2 = 1 \qquad \text{Substitute } \cos t = \tfrac{3}{5}$$
$$\sin^2 t = 1 - \tfrac{9}{25} = \tfrac{16}{25} \qquad \text{Solve for } \sin^2 t$$
$$\sin t = \pm\tfrac{4}{5} \qquad \text{Take square roots}$$

Since this point is in Quadrant IV, $\sin t$ is negative, so $\sin t = -\frac{4}{5}$. Now that we know both $\sin t$ and $\cos t$, we can find the values of the other trigonometric functions using the reciprocal identities.

$$\sin t = -\frac{4}{5} \qquad \cos t = \frac{3}{5} \qquad \tan t = \frac{\sin t}{\cos t} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$

$$\csc t = \frac{1}{\sin t} = -\frac{5}{4} \qquad \sec t = \frac{1}{\cos t} = \frac{5}{3} \qquad \cot t = \frac{1}{\tan t} = -\frac{3}{4}$$

Now Try Exercise 63 ■

EXAMPLE 6 ■ Writing One Trigonometric Function in Terms of Another

Write $\tan t$ in terms of $\cos t$, where t is in Quadrant III.

SOLUTION Since $\tan t = \sin t/\cos t$, we need to write $\sin t$ in terms of $\cos t$. By the Pythagorean identities we have

$$\sin^2 t + \cos^2 t = 1$$
$$\sin^2 t = 1 - \cos^2 t \qquad \text{Solve for } \sin^2 t$$
$$\sin t = \pm\sqrt{1 - \cos^2 t} \qquad \text{Take square roots}$$

Since $\sin t$ is negative in Quadrant III, the negative sign applies here. Thus

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\sqrt{1 - \cos^2 t}}{\cos t}$$

Now Try Exercise 53 ■

5.2 EXERCISES

CONCEPTS

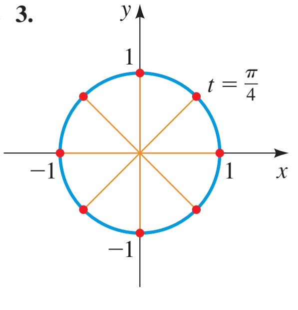
1. Let $P(x, y)$ be the terminal point on the unit circle determined by t. Then $\sin t =$ ________, $\cos t =$ ________, and $\tan t =$ ________.

2. If $P(x, y)$ is on the unit circle, then $x^2 + y^2 =$ ________. So for all t we have $\sin^2 t + \cos^2 t =$ ________.

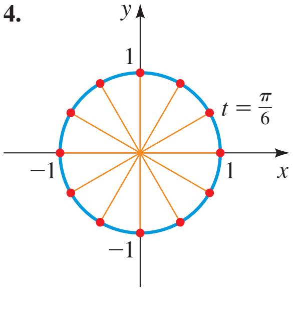
SKILLS

3–4 ■ Evaluating Trigonometric Functions Find $\sin t$ and $\cos t$ for the values of t whose terminal points are shown on the unit circle in the figure. In Exercise 3, t increases in increments of $\pi/4$; in Exercise 4, t increases in increments of $\pi/6$. (See Exercises 21 and 22 in Section 5.1.)

3.



4.



5–22 ■ Evaluating Trigonometric Functions Find the exact value of the trigonometric function at the given real number.

5. **(a)** $\sin\frac{7\pi}{6}$ **(b)** $\cos\frac{17\pi}{6}$ **(c)** $\tan\frac{7\pi}{6}$

6. **(a)** $\sin\frac{5\pi}{3}$ **(b)** $\cos\frac{11\pi}{3}$ **(c)** $\tan\frac{5\pi}{3}$

7. **(a)** $\sin\frac{11\pi}{4}$ **(b)** $\sin\left(-\frac{\pi}{4}\right)$ **(c)** $\sin\frac{5\pi}{4}$

8. **(a)** $\cos\frac{19\pi}{6}$ **(b)** $\cos\left(-\frac{7\pi}{6}\right)$ **(c)** $\cos\left(-\frac{\pi}{6}\right)$

9. **(a)** $\cos\frac{3\pi}{4}$ **(b)** $\cos\frac{5\pi}{4}$ **(c)** $\cos\frac{7\pi}{4}$

10. **(a)** $\sin\frac{3\pi}{4}$ **(b)** $\sin\frac{5\pi}{4}$ **(c)** $\sin\frac{7\pi}{4}$

11. **(a)** $\sin\frac{7\pi}{3}$ **(b)** $\csc\frac{7\pi}{3}$ **(c)** $\cot\frac{7\pi}{3}$

12. **(a)** $\csc\frac{5\pi}{4}$ **(b)** $\sec\frac{5\pi}{4}$ **(c)** $\tan\frac{5\pi}{4}$

13. **(a)** $\cos\left(-\frac{\pi}{3}\right)$ **(b)** $\sec\left(-\frac{\pi}{3}\right)$ **(c)** $\sin\left(-\frac{\pi}{3}\right)$

14. **(a)** $\tan\left(-\frac{\pi}{4}\right)$ **(b)** $\csc\left(-\frac{\pi}{4}\right)$ **(c)** $\cot\left(-\frac{\pi}{4}\right)$

15. **(a)** $\cos\left(-\frac{\pi}{6}\right)$ **(b)** $\csc\left(-\frac{\pi}{3}\right)$ **(c)** $\tan\left(-\frac{\pi}{6}\right)$

16. **(a)** $\sin\left(-\frac{\pi}{4}\right)$ **(b)** $\sec\left(-\frac{\pi}{4}\right)$ **(c)** $\cot\left(-\frac{\pi}{6}\right)$

17. **(a)** $\csc\frac{7\pi}{6}$ **(b)** $\sec\left(-\frac{\pi}{6}\right)$ **(c)** $\cot\left(-\frac{5\pi}{6}\right)$

18. **(a)** $\sec\frac{3\pi}{4}$ **(b)** $\cos\left(-\frac{2\pi}{3}\right)$ **(c)** $\tan\left(-\frac{7\pi}{6}\right)$

19. **(a)** $\sin\frac{4\pi}{3}$ **(b)** $\sec\frac{11\pi}{6}$ **(c)** $\cot\left(-\frac{\pi}{3}\right)$

20. **(a)** $\csc\frac{2\pi}{3}$ **(b)** $\sec\left(-\frac{5\pi}{3}\right)$ **(c)** $\cos\left(\frac{10\pi}{3}\right)$

21. **(a)** $\sin 13\pi$ **(b)** $\cos 14\pi$ **(c)** $\tan 15\pi$

22. **(a)** $\sin\frac{25\pi}{2}$ **(b)** $\cos\frac{25\pi}{2}$ **(c)** $\cot\frac{25\pi}{2}$

23–26 ■ Evaluating Trigonometric Functions Find the value of each of the six trigonometric functions (if it is defined) at the given real number t. Use your answers to complete the table.

23. $t = 0$ **24.** $t = \frac{\pi}{2}$

25. $t = \pi$ **26.** $t = \frac{3\pi}{2}$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1		undefined		
$\frac{\pi}{2}$						
π			0			undefined
$\frac{3\pi}{2}$						

27–36 ■ Evaluating Trigonometric Functions The terminal point $P(x, y)$ determined by a real number t is given. Find $\sin t$, $\cos t$, and $\tan t$.

27. $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ **28.** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

29. $\left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$ **30.** $\left(\frac{1}{5}, -\frac{2\sqrt{6}}{5}\right)$

31. $\left(-\frac{6}{7}, \frac{\sqrt{13}}{7}\right)$ **32.** $\left(\frac{40}{41}, \frac{9}{41}\right)$

33. $\left(-\frac{5}{13}, -\frac{12}{13}\right)$ **34.** $\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$

35. $\left(-\frac{20}{29}, \frac{21}{29}\right)$ **36.** $\left(\frac{24}{25}, -\frac{7}{25}\right)$

37–44 ■ Values of Trigonometric Functions Find an approximate value of the given trigonometric function by using **(a)** the figure and **(b)** a calculator. Compare the two values.

37. $\sin 1$

38. $\cos 0.8$

39. $\sin 1.2$

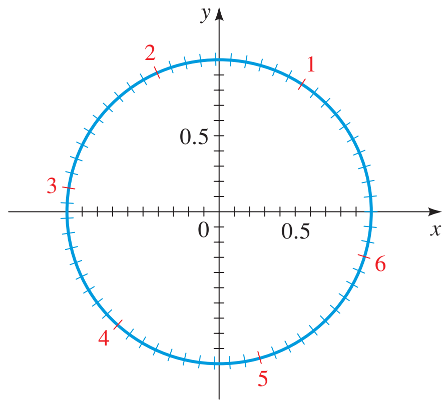
40. $\cos 5$

41. $\tan 0.8$

42. $\tan(-1.3)$

43. $\cos 4.1$

44. $\sin(-5.2)$



45–48 ■ Sign of a Trigonometric Expression Find the sign of the expression if the terminal point determined by t is in the given quadrant.

45. $\sin t \cos t$, Quadrant II **46.** $\tan t \sec t$, Quadrant IV

47. $\frac{\tan t \sin t}{\cot t}$, Quadrant III **48.** $\cos t \sec t$, any quadrant

49–52 ■ Quadrant of a Terminal Point From the information given, find the quadrant in which the terminal point determined by t lies.

49. $\sin t > 0$ and $\cos t < 0$

50. $\tan t > 0$ and $\sin t < 0$

51. $\csc t > 0$ and $\sec t < 0$

52. $\cos t < 0$ and $\cot t < 0$

53–62 ■ Writing One Trigonometric Expression in Terms of Another Write the first expression in terms of the second if the terminal point determined by t is in the given quadrant.

53. $\sin t$, $\cos t$; Quadrant II **54.** $\cos t$, $\sin t$; Quadrant IV

55. $\tan t$, $\sin t$; Quadrant IV **56.** $\tan t$, $\cos t$; Quadrant III

57. $\sec t$, $\tan t$; Quadrant II **58.** $\csc t$, $\cot t$; Quadrant III

59. $\tan t$, $\sec t$; Quadrant III **60.** $\sin t$, $\sec t$; Quadrant IV

61. $\tan^2 t$, $\sin t$; any quadrant

62. $\sec^2 t \sin^2 t$, $\cos t$; any quadrant

63–70 ■ Using the Pythagorean Identities Find the values of the trigonometric functions of t from the given information.

63. $\sin t = -\frac{4}{5}$, terminal point of t is in Quadrant IV

64. $\cos t = -\frac{7}{25}$, terminal point of t is in Quadrant III

65. $\sec t = 3$, terminal point of t is in Quadrant IV

66. $\tan t = \frac{1}{4}$, terminal point of t is in Quadrant III

67. $\tan t = -\frac{12}{5}$, $\sin t > 0$

68. $\csc t = 5$, $\cos t < 0$

69. $\sin t = -\frac{1}{4}$, $\sec t < 0$

70. $\tan t = -4$, $\csc t > 0$

SKILLS Plus

71–78 ■ Even and Odd Functions Determine whether the function is even, odd, or neither. (See page 204 for the definitions of even and odd functions.)

71. $f(x) = x^2 \sin x$ **72.** $f(x) = x^2 \cos 2x$

73. $f(x) = \sin x \cos x$ **74.** $f(x) = \sin x + \cos x$

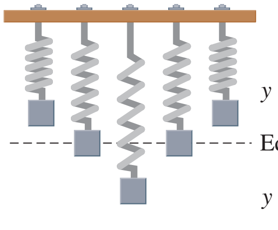
75. $f(x) = |x| \cos x$ **76.** $f(x) = x \sin^3 x$

77. $f(x) = x^3 + \cos x$ **78.** $f(x) = \cos(\sin x)$

APPLICATIONS

79. Harmonic Motion The displacement from equilibrium of an oscillating mass attached to a spring is given by $y(t) = 4 \cos 3\pi t$ where y is measured in inches and t in seconds. Find the displacement at the times indicated in the table.

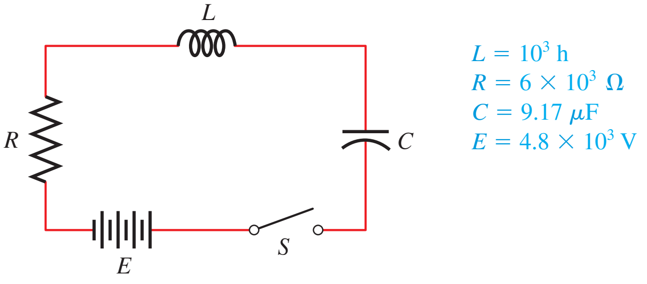
t	$y(t)$
0	
0.25	
0.50	
0.75	
1.00	
1.25	



80. Circadian Rhythms Everybody's blood pressure varies over the course of the day. In a certain individual the resting diastolic blood pressure at time t is given by $B(t) = 80 + 7 \sin(\pi t/12)$, where t is measured in hours since midnight and $B(t)$ in mmHg (millimeters of mercury). Find this person's resting diastolic blood pressure at

(a) 6:00 A.M. **(b)** 10:30 A.M. **(c)** Noon **(d)** 8:00 P.M.

81. Electric Circuit After the switch is closed in the circuit shown, the current t seconds later is $I(t) = 0.8e^{-3t} \sin 10t$. Find the current at the times **(a)** $t = 0.1$ s and **(b)** $t = 0.5$ s.



82. Bungee Jumping A bungee jumper plummets from a high bridge to the river below and then bounces back over and over again. At time t seconds after her jump, her height H (in meters) above the river is given by $H(t) = 100 + 75e^{-t/20}\cos\left(\frac{\pi}{4}t\right)$. Find her height at the times indicated in the table.

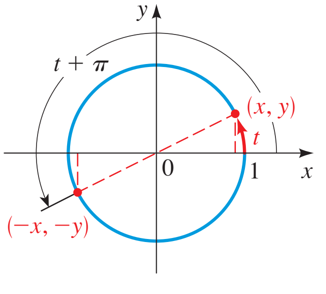
t	$H(t)$
0	
1	
2	
4	
6	
8	
12	

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

83. DISCOVER ■ PROVE: Reduction Formulas A *reduction formula* is one that can be used to "reduce" the number of terms in the input for a trigonometric function. Explain how the figure shows that the following reduction formulas are valid:

$$\sin(t + \pi) = -\sin t \qquad \cos(t + \pi) = -\cos t$$

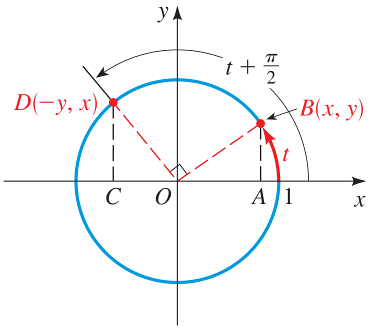
$$\tan(t + \pi) = \tan t$$



84. DISCOVER ■ PROVE: More Reduction Formulas By the Angle-Side-Angle Theorem from elementary geometry, triangles CDO and AOB in the figure to the right are congruent. Explain how this proves that if B has coordinates (x, y), then D has coordinates $(-y, x)$. Then explain how the figure shows that the following reduction formulas are valid:

$$\sin\left(t + \frac{\pi}{2}\right) = \cos t \qquad \cos\left(t + \frac{\pi}{2}\right) = -\sin t$$

$$\tan\left(t + \frac{\pi}{2}\right) = -\cot t$$



5.3 TRIGONOMETRIC GRAPHS

■ Graphs of Sine and Cosine ■ Graphs of Transformations of Sine and Cosine
■ Using Graphing Devices to Graph Trigonometric Functions

The graph of a function gives us a better idea of its behavior. So in this section we graph the sine and cosine functions and certain transformations of these functions. The other trigonometric functions are graphed in the next section.

■ Graphs of Sine and Cosine

To help us graph the sine and cosine functions, we first observe that these functions repeat their values in a regular fashion. To see exactly how this happens, recall that the circumference of the unit circle is 2π. It follows that the terminal point $P(x, y)$ determined by the real number t is the same as that determined by $t + 2\pi$. Since the sine and cosine functions are defined in terms of the coordinates of $P(x, y)$, it follows that their values are unchanged by the addition of any integer multiple of 2π. In other words,

$$\sin(t + 2n\pi) = \sin t \qquad \text{for any integer } n$$

$$\cos(t + 2n\pi) = \cos t \qquad \text{for any integer } n$$

Thus the sine and cosine functions are *periodic* according to the following definition: A function f is **periodic** if there is a positive number p such that $f(t + p) = f(t)$ for every t. The least such positive number (if it exists) is the **period** of f. If f has period p, then the graph of f on any interval of length p is called **one complete period** of f.

PERIODIC PROPERTIES OF SINE AND COSINE

The functions sine and cosine have period 2π:

$$\sin(t + 2\pi) = \sin t \qquad \cos(t + 2\pi) = \cos t$$

TABLE 1

t	$\sin t$	$\cos t$
$0 \to \frac{\pi}{2}$	$0 \to 1$	$1 \to 0$
$\frac{\pi}{2} \to \pi$	$1 \to 0$	$0 \to -1$
$\pi \to \frac{3\pi}{2}$	$0 \to -1$	$-1 \to 0$
$\frac{3\pi}{2} \to 2\pi$	$-1 \to 0$	$0 \to 1$

So the sine and cosine functions repeat their values in any interval of length 2π. To sketch their graphs, we first graph one period. To sketch the graphs on the interval $0 \le t \le 2\pi$, we could try to make a table of values and use those points to draw the graph. Since no such table can be complete, let's look more closely at the definitions of these functions.

Recall that $\sin t$ is the y-coordinate of the terminal point $P(x, y)$ on the unit circle determined by the real number t. How does the y-coordinate of this point vary as t increases? It's easy to see that the y-coordinate of $P(x, y)$ increases to 1, then decreases to -1 repeatedly as the point $P(x, y)$ travels around the unit circle. (See Figure 1.) In fact, as t increases from 0 to $\pi/2$, $y = \sin t$ increases from 0 to 1. As t increases from $\pi/2$ to π, the value of $y = \sin t$ decreases from 1 to 0. Table 1 shows the variation of the sine and cosine functions for t between 0 and 2π.

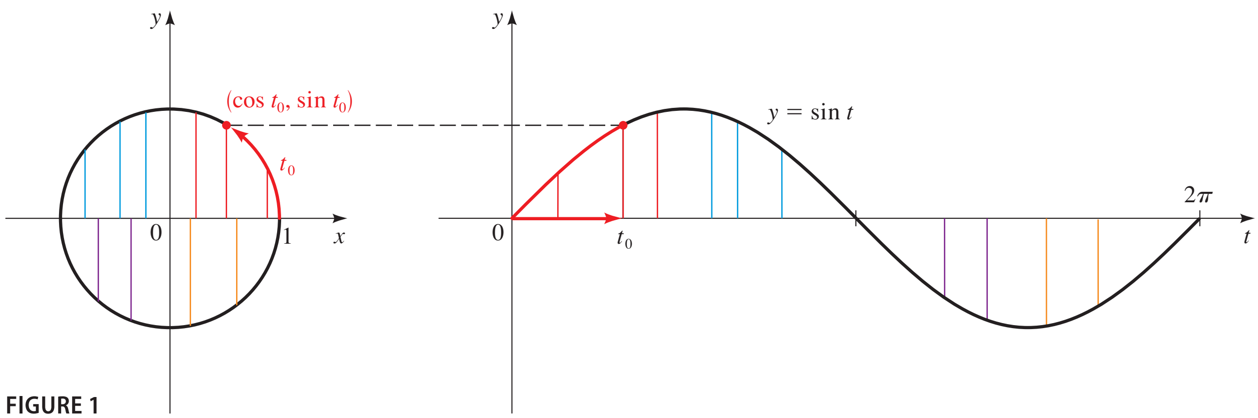


FIGURE 1

To draw the graphs more accurately, we find a few other values of $\sin t$ and $\cos t$ in Table 2. We could find still other values with the aid of a calculator.

TABLE 2

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

Now we use this information to graph the functions $\sin t$ and $\cos t$ for t between 0 and 2π in Figures 2 and 3. These are the graphs of one period. Using the fact that these functions are periodic with period 2π, we get their complete graphs by continuing the same pattern to the left and to the right in every successive interval of length 2π.

The graph of the sine function is symmetric with respect to the origin. This is as expected, since sine is an odd function. Since the cosine function is an even function, its graph is symmetric with respect to the y-axis.

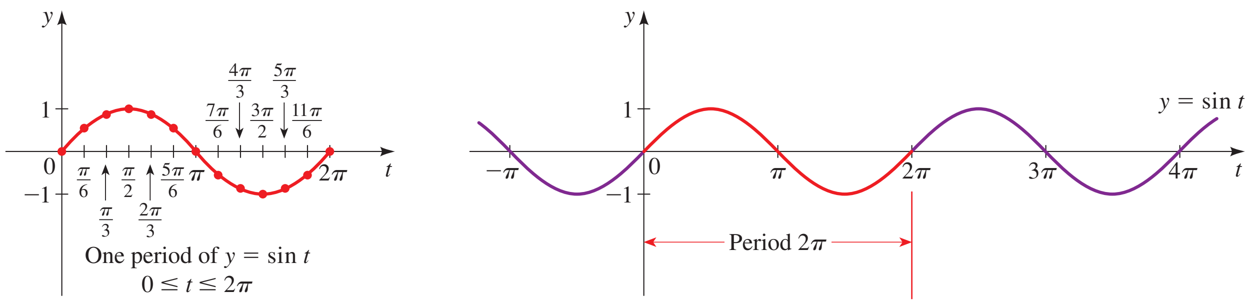


FIGURE 2 Graph of $\sin t$

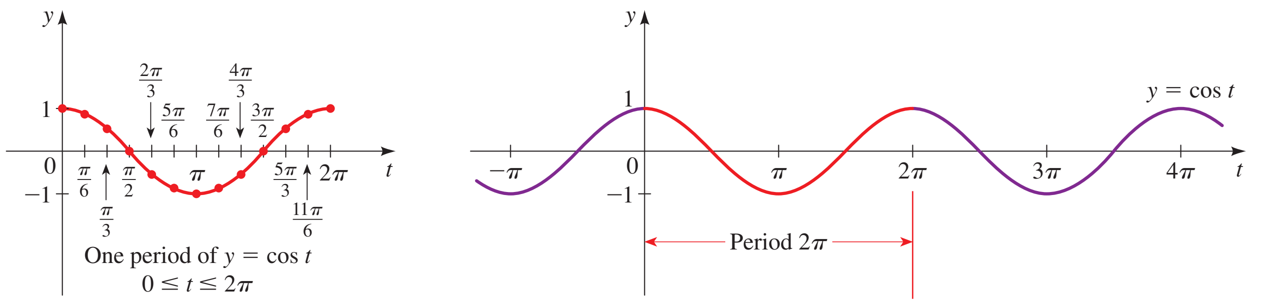


FIGURE 3 Graph of cos t

Graphs of Transformations of Sine and Cosine

We now consider graphs of functions that are transformations of the sine and cosine functions. Thus, the graphing techniques of Section 2.6 are very useful here. The graphs we obtain are important for understanding applications to physical situations such as harmonic motion (see Section 5.6), but some of them are beautiful graphs that are interesting in their own right.

It's traditional to use the letter x to denote the variable in the domain of a function. So from here on we use the letter x and write $y = \sin x$, $y = \cos x$, $y = \tan x$, and so on to denote these functions.

EXAMPLE 1 ■ Cosine Curves

Sketch the graph of each function.

(a) $f(x) = 2 + \cos x$ **(b)** $g(x) = -\cos x$

SOLUTION

(a) The graph of $y = 2 + \cos x$ is the same as the graph of $y = \cos x$, but shifted up 2 units (see Figure 4(a)).

(b) The graph of $y = -\cos x$ in Figure 4(b) is the reflection of the graph of $y = \cos x$ in the x-axis.

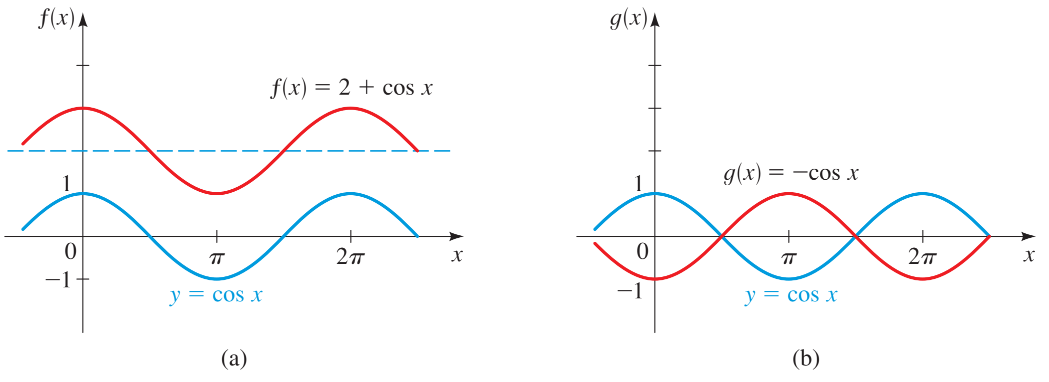


FIGURE 4

Now Try Exercises 5 and 7 ■

Let's graph $y = 2\sin x$. We start with the graph of $y = \sin x$ and multiply the y-coordinate of each point by 2. This has the effect of stretching the graph vertically by a factor of 2. To graph $y = \frac{1}{2}\sin x$, we start with the graph of $y = \sin x$ and multiply

Vertical stretching and shrinking of graphs is discussed in Section 2.6.

the y-coordinate of each point by $\frac{1}{2}$. This has the effect of shrinking the graph vertically by a factor of $\frac{1}{2}$ (see Figure 5).

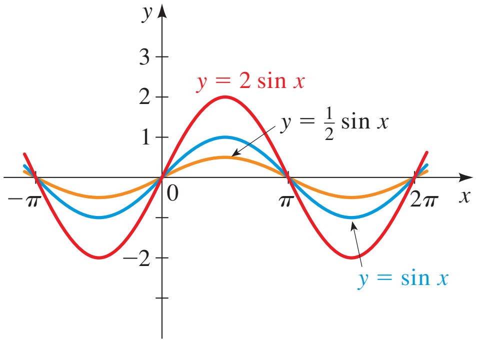


FIGURE 5

In general, for the functions

$$y = a \sin x \qquad \text{and} \qquad y = a \cos x$$

the number $|a|$ is called the **amplitude** and is the largest value these functions attain. Graphs of $y = a \sin x$ for several values of a are shown in Figure 6.

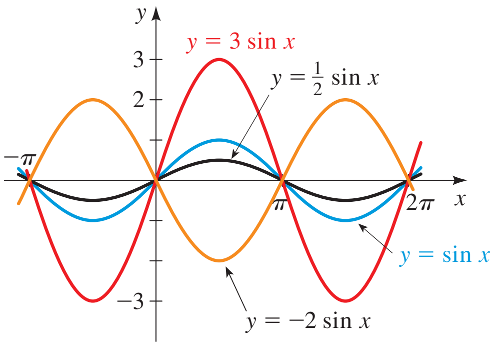


FIGURE 6

EXAMPLE 2 ■ Stretching a Cosine Curve

Find the amplitude of $y = -3 \cos x$, and sketch its graph.

SOLUTION The amplitude is $|-3| = 3$, so the largest value the graph attains is 3 and the smallest value is -3. To sketch the graph, we begin with the graph of $y = \cos x$, stretch the graph vertically by a factor of 3, and reflect in the x-axis, arriving at the graph in Figure 7.

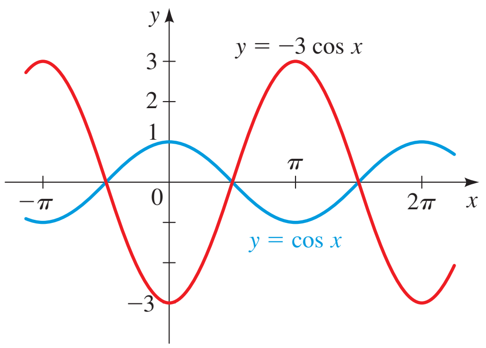


FIGURE 7

Now Try Exercise 11 ■

Since the sine and cosine functions have period 2π, the functions

$$y = a \sin kx \qquad \text{and} \qquad y = a \cos kx \qquad (k > 0)$$

complete one period as kx varies from 0 to 2π, that is, for $0 \le kx \le 2\pi$ or for $0 \le x \le 2\pi/k$. So these functions complete one period as x varies between 0 and $2\pi/k$ and thus have period $2\pi/k$. The graphs of these functions are called **sine curves** and **cosine curves**, respectively. (Collectively, sine and cosine curves are often referred to as **sinusoidal** curves.)

SINE AND COSINE CURVES

The sine and cosine curves

$$y = a \sin kx \qquad \text{and} \qquad y = a \cos kx \qquad (k > 0)$$

have **amplitude** $|a|$ and **period** $2\pi/k$.

An appropriate interval on which to graph one complete period is $[0, 2\pi/k]$.

Horizontal stretching and shrinking of graphs is discussed in Section 2.6.

To see how the value of k affects the graph of $y = \sin kx$, let's graph the sine curve $y = \sin 2x$. Since the period is $2\pi/2 = \pi$, the graph completes one period in the interval $0 \le x \le \pi$ (see Figure 8(a)). For the sine curve $y = \sin \frac{1}{2}x$ the period is $2\pi \div \frac{1}{2} = 4\pi$, so the graph completes one period in the interval $0 \le x \le 4\pi$ (see Figure 8(b)). We see that the effect is to *shrink* the graph horizontally if $k > 1$ or to *stretch* the graph horizontally if $k < 1$.

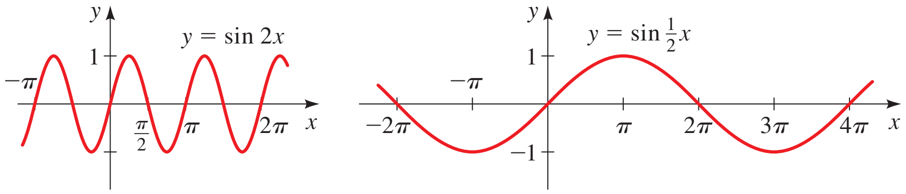


FIGURE 8 (a) (b)

For comparison, in Figure 9 we show the graphs of one period of the sine curve $y = a \sin kx$ for several values of k.

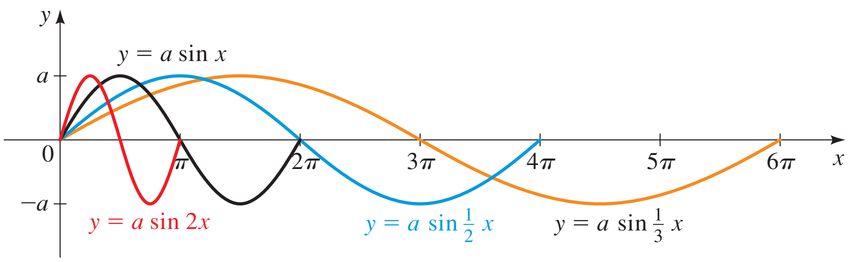


FIGURE 9

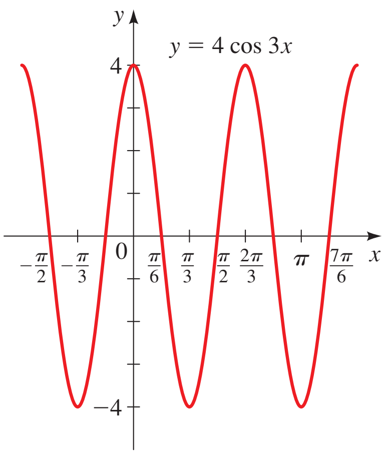


FIGURE 10

EXAMPLE 3 ■ Amplitude and Period

Find the amplitude and period of each function, and sketch its graph.

(a) $y = 4 \cos 3x$ **(b)** $y = -2 \sin \frac{1}{2}x$

SOLUTION

(a) We get the amplitude and period from the form of the function as follows.

$$\text{amplitude} = |a| = 4$$

$$y = 4 \cos 3x$$

$$\text{period} = \frac{2\pi}{k} = \frac{2\pi}{3}$$

The amplitude is 4, and the period is $2\pi/3$. The graph is shown in Figure 10.

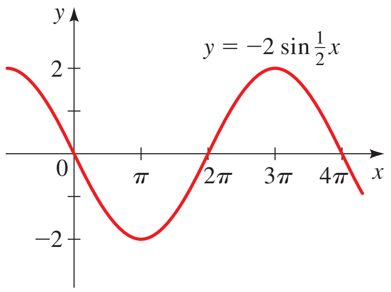


FIGURE 11

(b) For $y = -2\sin\frac{1}{2}x$,

$$\text{amplitude} = |a| = |-2| = 2$$

$$\text{period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

The graph is shown in Figure 11.

Now Try Exercises 23 and 25 ■

The graphs of functions of the form $y = a\sin k(x - b)$ and $y = a\cos k(x - b)$ are simply sine and cosine curves shifted horizontally by an amount $|b|$. They are shifted to the right if $b > 0$ or to the left if $b < 0$. We summarize the properties of these functions in the following box.

The phase shift of a sine curve is discussed in Section 5.6.

SHIFTED SINE AND COSINE CURVES

The sine and cosine curves

$$y = a\sin k(x - b) \qquad \text{and} \qquad y = a\cos k(x - b) \qquad (k > 0)$$

have **amplitude** $|a|$, **period** $2\pi/k$, and **horizontal shift** b.

An appropriate interval on which to graph one complete period is $[b, b + (2\pi/k)]$.

The graphs of $y = \sin\left(x - \frac{\pi}{3}\right)$ and $y = \sin\left(x + \frac{\pi}{6}\right)$ are shown in Figure 12.

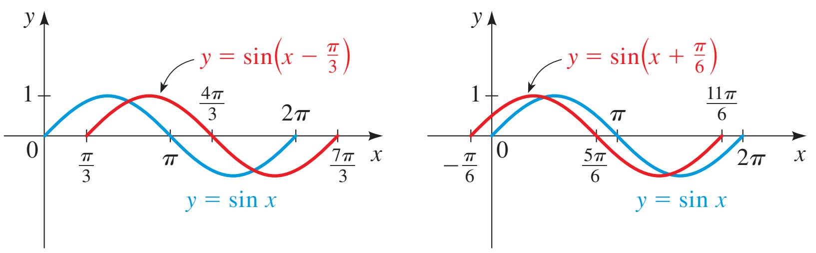


FIGURE 12 Horizontal shifts of a sine curve

EXAMPLE 4 ■ A Horizontally Shifted Sine Curve

Find the amplitude, period, and horizontal shift of $y = 3\sin 2\left(x - \frac{\pi}{4}\right)$, and graph one complete period.

SOLUTION We get the amplitude, period, and horizontal shift from the form of the function as follows:

$$\text{amplitude} = |a| = 3 \qquad \text{period} = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi$$

$$y = 3\sin 2\left(x - \frac{\pi}{4}\right)$$

$$\text{horizontal shift} = \frac{\pi}{4} \text{ (to the right)}$$

Since the horizontal shift is $\pi/4$ and the period is π, one complete period occurs on the interval

$$\left[\frac{\pi}{4}, \frac{\pi}{4} + \pi\right] = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

Here is another way to find an appropriate interval on which to graph one complete period. Since the period of $y = \sin x$ is 2π, the function $y = 3\sin 2\left(x - \frac{\pi}{4}\right)$ will go through one complete period as $2\left(x - \frac{\pi}{4}\right)$ varies from 0 to 2π.

Start of period:	End of period:
$2\left(x - \frac{\pi}{4}\right) = 0$	$2\left(x - \frac{\pi}{4}\right) = 2\pi$
$x - \frac{\pi}{4} = 0$	$x - \frac{\pi}{4} = \pi$
$x = \frac{\pi}{4}$	$x = \frac{5\pi}{4}$

So we graph one period on the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

As an aid in sketching the graph, we divide this interval into four equal parts, then graph a sine curve with amplitude 3 as in Figure 13.

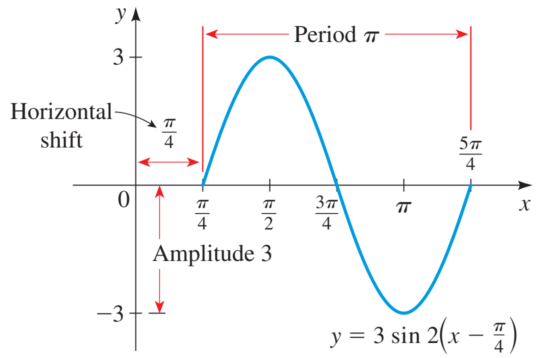


FIGURE 13

Now Try Exercise 35 ■

EXAMPLE 5 ■ A Horizontally Shifted Cosine Curve

Find the amplitude, period, and horizontal shift of $y = \frac{3}{4}\cos\left(2x + \frac{2\pi}{3}\right)$, and graph one complete period.

SOLUTION We first write this function in the form $y = a\cos k(x - b)$. To do this, we factor 2 from the expression $2x + \dfrac{2\pi}{3}$ to get

$$y = \frac{3}{4}\cos 2\left[x - \left(-\frac{\pi}{3}\right)\right]$$

Thus we have

$$\text{amplitude} = |a| = \frac{3}{4}$$

$$\text{period} = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi$$

$$\text{horizontal shift} = b = -\frac{\pi}{3} \qquad \text{Shift } \frac{\pi}{3} \text{ to the } \textit{left}$$

We can also find one complete period as follows:

Start of period:	End of period:
$2x + \frac{2\pi}{3} = 0$	$2x + \frac{2\pi}{3} = 2\pi$
$2x = -\frac{2\pi}{3}$	$2x = \frac{4\pi}{3}$
$x = -\frac{\pi}{3}$	$x = \frac{2\pi}{3}$

So we graph one period on the interval $\left[-\frac{\pi}{3}, \frac{2\pi}{3}\right]$.

From this information it follows that one period of this cosine curve begins at $-\pi/3$ and ends at $(-\pi/3) + \pi = 2\pi/3$. To sketch the graph over the interval $[-\pi/3, 2\pi/3]$, we divide this interval into four equal parts and graph a cosine curve with amplitude $\frac{3}{4}$ as shown in Figure 14.

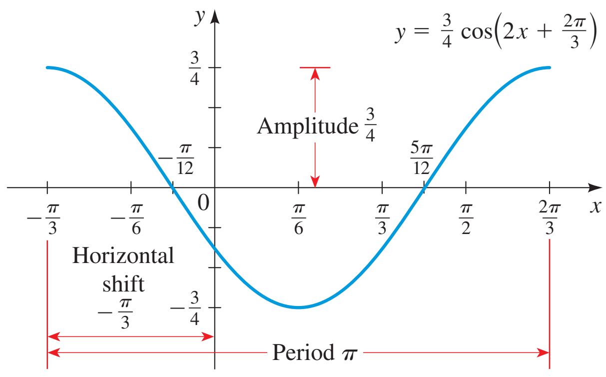


FIGURE 14

Now Try Exercise 37 ■

■ Using Graphing Devices to Graph Trigonometric Functions

See Appendix C, *Graphing with a Graphing Calculator,* for guidelines on choosing an appropriate viewing rectangle. Go to **www.stewartmath.com**.

When using a graphing calculator or a computer to graph a function, it is important to choose the viewing rectangle carefully in order to produce a reasonable graph of the function. This is especially true for trigonometric functions; the next example shows that, if care is not taken, it's easy to produce a very misleading graph of a trigonometric function.

EXAMPLE 6 ■ Choosing the Viewing Rectangle

Graph the function $f(x) = \sin 50x$ in an appropriate viewing rectangle.

SOLUTION Figure 15(a) shows the graph of f produced by a graphing calculator using the viewing rectangle $[-12, 12]$ by $[-1.5, 1.5]$. At first glance the graph appears to be reasonable. But if we change the viewing rectangle to the ones shown in Figure 15, the graphs look very different. Something strange is happening.

The appearance of the graphs in Figure 15 depends on the machine used. The graphs you get with your own graphing device might not look like these figures, but they will also be quite inaccurate.

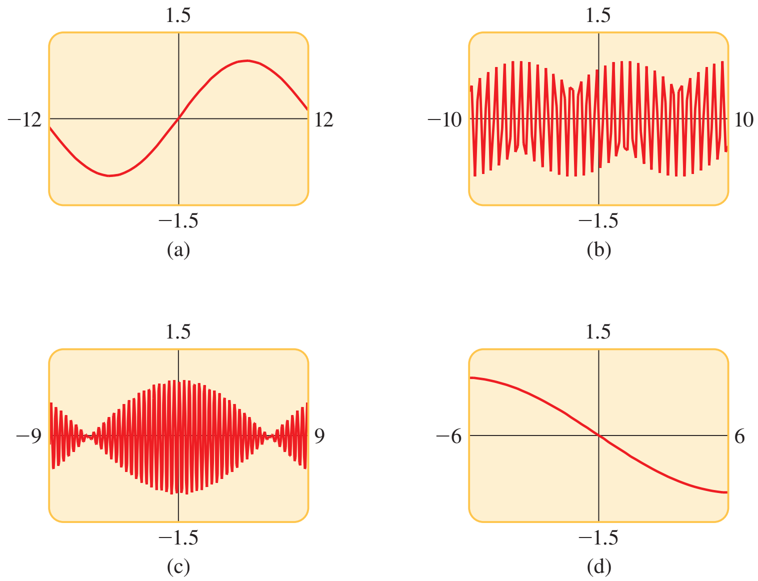


FIGURE 15 Graphs of $f(x) = \sin 50x$ in different viewing rectangles

To explain the big differences in appearance of these graphs and to find an appropriate viewing rectangle, we need to find the period of the function $y = \sin 50x$.

$$\text{period} = \frac{2\pi}{50} = \frac{\pi}{25} \approx 0.126$$

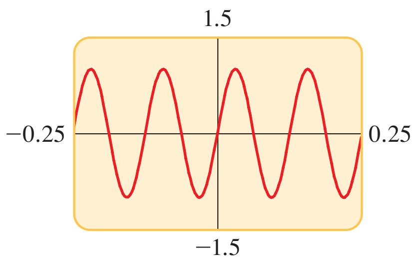


FIGURE 16 $f(x) = \sin 50x$

This suggests that we should deal only with small values of x in order to show just a few oscillations of the graph. If we choose the viewing rectangle $[-0.25, 0.25]$ by $[-1.5, 1.5]$, we get the graph shown in Figure 16.

Now we see what went wrong in Figure 15. The oscillations of $y = \sin 50x$ are so rapid that when the calculator plots points and joins them, it misses most of the maximum and minimum points and therefore gives a very misleading impression of the graph.

Now Try Exercise 55 ■

The function h in Example 7 is **periodic** with period 2π. In general, functions that are sums of functions from the following list are periodic:

$$1, \cos kx, \cos 2kx, \cos 3kx, \ldots$$

$$\sin kx, \sin 2kx, \sin 3kx, \ldots$$

Although these functions appear to be special, they are actually fundamental to describing all periodic functions that arise in practice. The French mathematician J. B. J. Fourier (see page 546) discovered that nearly every periodic function can be written as a sum (usually an infinite sum) of these functions. This is remarkable because it means that any situation in which periodic variation occurs can be described mathematically using the functions sine and cosine. A modern application of Fourier's discovery is the digital encoding of sound on compact discs.

EXAMPLE 7 ■ A Sum of Sine and Cosine Curves

Graph $f(x) = 2 \cos x$, $g(x) = \sin 2x$, and $h(x) = 2 \cos x + \sin 2x$ on a common screen to illustrate the method of graphical addition.

SOLUTION Notice that $h = f + g$, so its graph is obtained by adding the corresponding y-coordinates of the graphs of f and g. The graphs of f, g, and h are shown in Figure 17.

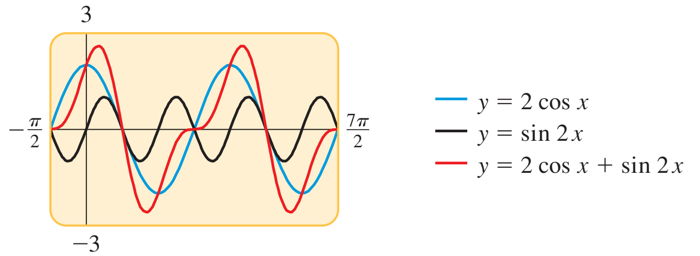


FIGURE 17

Now Try Exercise 63 ■

EXAMPLE 8 ■ A Cosine Curve with Variable Amplitude

Graph the functions $y = x^2$, $y = -x^2$, and $y = x^2 \cos 6\pi x$ on a common screen. Comment on and explain the relationship among the graphs.

SOLUTION Figure 18 shows all three graphs in the viewing rectangle $[-1.5, 1.5]$ by $[-2, 2]$. It appears that the graph of $y = x^2 \cos 6\pi x$ lies between the graphs of the functions $y = x^2$ and $y = -x^2$.

To understand this, recall that the values of $\cos 6\pi x$ lie between -1 and 1, that is,

$$-1 \le \cos 6\pi x \le 1$$

for all values of x. Multiplying the inequalities by x^2 and noting that $x^2 \ge 0$, we get

$$-x^2 \le x^2 \cos 6\pi x \le x^2$$

This explains why the functions $y = x^2$ and $y = -x^2$ form a boundary for the graph of $y = x^2 \cos 6\pi x$. (Note that the graphs touch when $\cos 6\pi x = \pm 1$.)

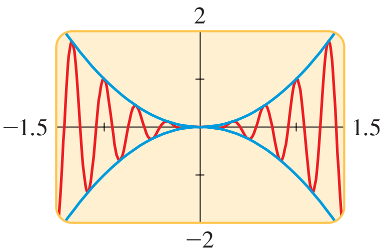


FIGURE 18 $y = x^2 \cos 6\pi x$

Now Try Exercise 69 ■

Example 8 shows that the function $y = x^2$ controls the amplitude of the graph of $y = x^2 \cos 6\pi x$. In general, if $f(x) = a(x) \sin kx$ or $f(x) = a(x) \cos kx$, the function a determines how the amplitude of f varies, and the graph of f lies between the graphs of $y = -a(x)$ and $y = a(x)$. Here is another example.

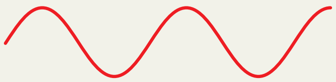
Jeffrey Lepore/Science Source

DISCOVERY PROJECT

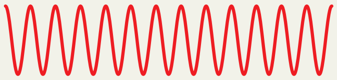
Predator/Prey Models

Many animal populations fluctuate regularly in size and so can be modeled by trigonometric functions Predicting population changes allows scientists to detect anomalies and take steps to protect a species. In this project we study the population of a predator species and the population of its prey. If the prey is abundant, the predator population grows, but too many predators tend to deplete the prey. This results in a decrease in the predator population, then the prey population increases, and so on. You can find the project at **www.stewartmath.com**.

AM and FM Radio
Radio transmissions consist of sound waves superimposed on a harmonic electromagnetic wave form called the **carrier signal**.

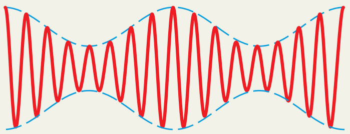


Sound wave



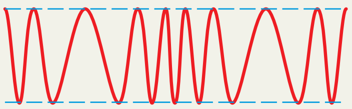
Carrier signal

There are two types of radio transmission, called **amplitude modulation (AM)** and **frequency modulation (FM)**. In AM broadcasting, the sound wave changes, or **modulates**, the amplitude of the carrier, but the frequency remains unchanged.



AM signal

In FM broadcasting, the sound wave modulates the frequency, but the amplitude remains the same.



FM signal

EXAMPLE 9 ■ A Cosine Curve with Variable Amplitude

Graph the function $f(x) = \cos 2\pi x \cos 16\pi x$.

SOLUTION The graph is shown in Figure 19. Although it was drawn by a computer, we could have drawn it by hand, by first sketching the boundary curves $y = \cos 2\pi x$ and $y = -\cos 2\pi x$. The graph of f is a cosine curve that lies between the graphs of these two functions.

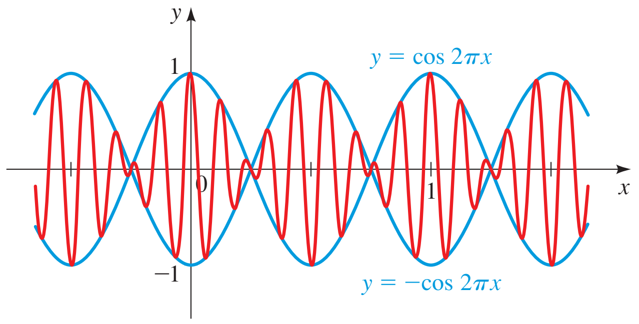


FIGURE 19 $f(x) = \cos 2\pi x \cos 16\pi x$

Now Try Exercise 71 ■

EXAMPLE 10 ■ A Sine Curve with Decaying Amplitude

The function $f(x) = \dfrac{\sin x}{x}$ is important in calculus. Graph this function, and comment on its behavior when x is close to 0.

SOLUTION The viewing rectangle $[-15, 15]$ by $[-0.5, 1.5]$ shown in Figure 20(a) gives a good global view of the graph of f. The viewing rectangle $[-1, 1]$ by $[-0.5, 1.5]$ in Figure 20(b) focuses on the behavior of f when $x \approx 0$. Notice that although $f(x)$ is not defined when $x = 0$ (in other words, 0 is not in the domain of f), the values of f seem to approach 1 when x gets close to 0. This fact is crucial in calculus.

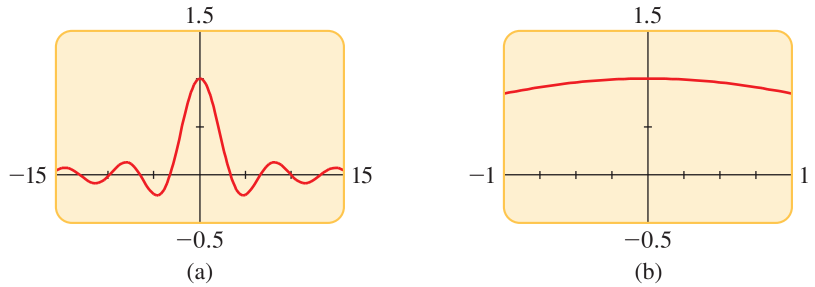


FIGURE 20 $f(x) = \dfrac{\sin x}{x}$

Now Try Exercise 81 ■

The function in Example 10 can be written as

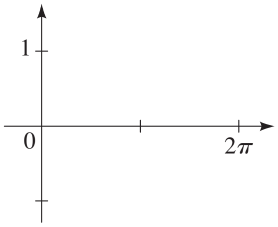
$$f(x) = \frac{1}{x} \sin x$$

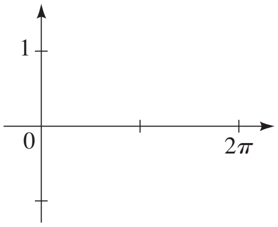
and may thus be viewed as a sine function whose amplitude is controlled by the function $a(x) = 1/x$.

5.3 EXERCISES

CONCEPTS

1. If a function f is periodic with period p, then $f(t + p) =$ ________ for every t. The trigonometric functions $y = \sin x$ and $y = \cos x$ are periodic, with period ________ and amplitude ________. Sketch a graph of each function on the interval $[0, 2\pi]$.





2. To obtain the graph of $y = 5 + \sin x$, we start with the graph of $y = \sin x$, then shift it 5 units ________ (upward/downward). To obtain the graph of $y = -\cos x$, we start with the graph of $y = \cos x$, then reflect it in the ____-axis.

3. The sine and cosine curves $y = a \sin kx$ and $y = a \cos kx$, $k > 0$, have amplitude ________ and period ________. The sine curve $y = 3 \sin 2x$ has amplitude ________ and period ________.

4. The sine curve $y = a \sin k(x - b)$ has amplitude ________, period ________, and horizontal shift ________. The sine curve $y = 4 \sin 3\left(x - \frac{\pi}{6}\right)$ has amplitude ________, period ________, and horizontal shift ________.

SKILLS

5–18 ■ Graphing Sine and Cosine Functions Graph the function.

5. $f(x) = 2 + \sin x$ **6.** $f(x) = -2 + \cos x$

7. $f(x) = -\sin x$ **8.** $f(x) = 2 - \cos x$

9. $f(x) = -2 + \sin x$ **10.** $f(x) = -1 + \cos x$

11. $g(x) = 3 \cos x$ **12.** $g(x) = 2 \sin x$

13. $g(x) = -\frac{1}{2} \sin x$ **14.** $g(x) = -\frac{2}{3} \cos x$

15. $g(x) = 3 + 3 \cos x$ **16.** $g(x) = 4 - 2 \sin x$

17. $h(x) = |\cos x|$ **18.** $h(x) = |\sin x|$

19–32 ■ Amplitude and Period Find the amplitude and period of the function, and sketch its graph.

19. $y = \cos 2x$ **20.** $y = -\sin 2x$

21. $y = -\sin 3x$ **22.** $y = \cos 4\pi x$

23. $y = -2 \cos 3\pi x$ **24.** $y = -3 \sin 6x$

25. $y = 10 \sin \frac{1}{2}x$ **26.** $y = 5 \cos \frac{1}{4}x$

27. $y = -\frac{1}{3} \cos \frac{1}{3}x$ **28.** $y = 4 \sin(-2x)$

29. $y = -2 \sin 2\pi x$ **30.** $y = -3 \sin \pi x$

31. $y = 1 + \frac{1}{2} \cos \pi x$ **32.** $y = -2 + \cos 4\pi x$

33–46 ■ Horizontal Shifts Find the amplitude, period, and horizontal shift of the function, and graph one complete period.

33. $y = \cos\left(x - \dfrac{\pi}{2}\right)$ **34.** $y = 2 \sin\left(x - \dfrac{\pi}{3}\right)$

35. $y = -2 \sin\left(x - \dfrac{\pi}{6}\right)$ **36.** $y = 3 \cos\left(x + \dfrac{\pi}{4}\right)$

37. $y = -4 \sin 2\left(x + \dfrac{\pi}{2}\right)$ **38.** $y = \sin \dfrac{1}{2}\left(x + \dfrac{\pi}{4}\right)$

39. $y = 5 \cos\left(3x - \dfrac{\pi}{4}\right)$ **40.** $y = 2 \sin\left(\dfrac{2}{3}x - \dfrac{\pi}{6}\right)$

41. $y = \dfrac{1}{2} - \dfrac{1}{2} \cos\left(2x - \dfrac{\pi}{3}\right)$ **42.** $y = 1 + \cos\left(3x + \dfrac{\pi}{2}\right)$

43. $y = 3 \cos \pi\left(x + \frac{1}{2}\right)$ **44.** $y = 3 + 2 \sin 3(x + 1)$

45. $y = \sin(\pi + 3x)$ **46.** $y = \cos\left(\dfrac{\pi}{2} - x\right)$

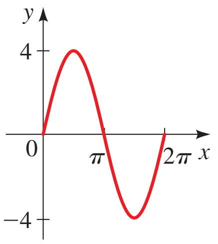
47–54 ■ Equations from a Graph The graph of one complete period of a sine or cosine curve is given.

(a) Find the amplitude, period, and horizontal shift.

(b) Write an equation that represents the curve in the form

$$y = a \sin k(x - b) \quad \text{or} \quad y = a \cos k(x - b)$$

47.



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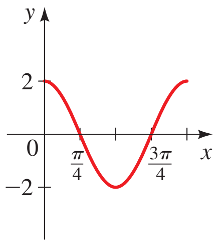

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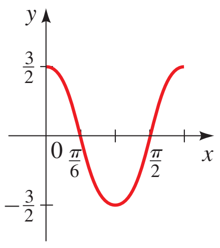

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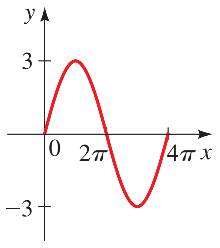

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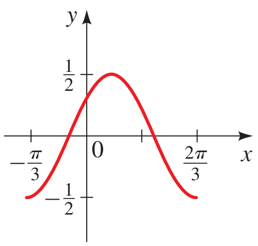

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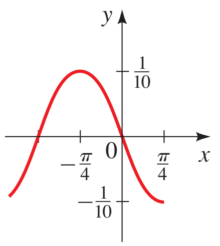

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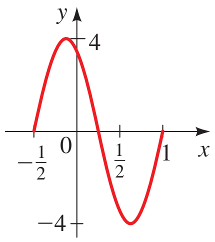

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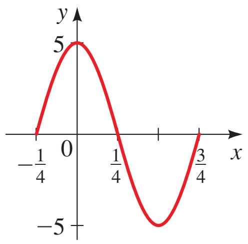


55–62 ■ Graphing Trigonometric Functions Determine an appropriate viewing rectangle for each function, and use it to draw the graph.

55. $f(x) = \cos 100x$

56. $f(x) = 3 \sin 120x$

57. $f(x) = \sin(x/40)$

58. $f(x) = \cos(x/80)$

59. $y = \tan 25x$

60. $y = \csc 40x$

61. $y = \sin^2 20x$

62. $y = \sqrt{\tan 10\pi x}$

63–66 ■ Graphical Addition Graph f, g, and $f + g$ on a common screen to illustrate graphical addition.

63. $f(x) = x, \quad g(x) = \sin x$

64. $f(x) = \sin x, \quad g(x) = \sin 2x$

65. $f(x) = \sin 3x, \quad g(x) = \cos \frac{1}{2}x$

66. $f(x) = 0.5 \sin 5x, \quad g(x) = -\cos 2x$

67–72 ■ Sine and Cosine Curves with Variable Amplitude Graph the three functions on a common screen. How are the graphs related?

67. $y = x^2, \quad y = -x^2, \quad y = x^2 \sin x$

68. $y = x, \quad y = -x, \quad y = x \cos x$

69. $y = \sqrt{x}, \quad y = -\sqrt{x}, \quad y = \sqrt{x} \sin 5\pi x$

70. $y = \dfrac{1}{1 + x^2}, \quad y = -\dfrac{1}{1 + x^2}, \quad y = \dfrac{\cos 2\pi x}{1 + x^2}$

71. $y = \cos 3\pi x, \quad y = -\cos 3\pi x, \quad y = \cos 3\pi x \cos 21\pi x$

72. $y = \sin 2\pi x, \quad y = -\sin 2\pi x, \quad y = \sin 2\pi x \sin 10\pi x$

SKILLS Plus

73–76 ■ Maxima and Minima Find the maximum and minimum values of the function.

73. $y = \sin x + \sin 2x$

74. $y = x - 2 \sin x, \quad 0 \le x \le 2\pi$

75. $y = 2 \sin x + \sin^2 x$

76. $y = \dfrac{\cos x}{2 + \sin x}$

77–80 ■ Solving Trigonometric Equations Graphically Find all solutions of the equation that lie in the interval $[0, \pi]$. State each answer rounded to two decimal places.

77. $\cos x = 0.4$

78. $\tan x = 2$

79. $\csc x = 3$

80. $\cos x = x$

81–82 ■ Limiting Behavior of Trigonometric Functions A function f is given.

(a) Is f even, odd, or neither?
(b) Find the x-intercepts of the graph of f.
(c) Graph f in an appropriate viewing rectangle.
(d) Describe the behavior of the function as $x \to \pm\infty$.
(e) Notice that $f(x)$ is not defined when $x = 0$. What happens as x approaches 0?

81. $f(x) = \dfrac{1 - \cos x}{x}$

82. $f(x) = \dfrac{\sin 4x}{2x}$

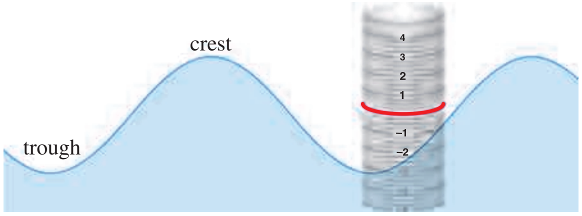
APPLICATIONS

83. Height of a Wave As a wave passes by an offshore piling, the height of the water is modeled by the function

$$h(t) = 3 \cos\left(\frac{\pi}{10}t\right)$$

where $h(t)$ is the height in feet above mean sea level at time t seconds.

(a) Find the period of the wave.
(b) Find the wave height, that is, the vertical distance between the trough and the crest of the wave.



84. Sound Vibrations A tuning fork is struck, producing a pure tone as its tines vibrate. The vibrations are modeled by the function

$$v(t) = 0.7 \sin(880\pi t)$$

where $v(t)$ is the displacement of the tines in millimeters at time t seconds.

(a) Find the period of the vibration.
(b) Find the frequency of the vibration, that is, the number of times the fork vibrates per second.
(c) Graph the function v.

85. Blood Pressure Each time your heart beats, your blood pressure first increases and then decreases as the heart rests between beats. The maximum and minimum blood pressures are called the *systolic* and *diastolic* pressures, respectively. Your *blood pressure reading* is written as systolic/diastolic. A reading of 120/80 is considered normal.

A certain person's blood pressure is modeled by the function

$$p(t) = 115 + 25 \sin(160\pi t)$$

where $p(t)$ is the pressure in mmHg (millimeters of mercury), at time t measured in minutes.

(a) Find the period of p.
(b) Find the number of heartbeats per minute.
(c) Graph the function p.
(d) Find the blood pressure reading. How does this compare to normal blood pressure?

86. Variable Stars Variable stars are ones whose brightness varies periodically. One of the most visible is R Leonis; its brightness is modeled by the function

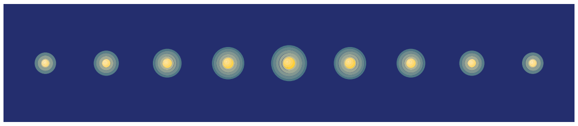
$$b(t) = 7.9 - 2.1\cos\left(\frac{\pi}{156}t\right)$$

where t is measured in days.

(a) Find the period of R Leonis.

(b) Find the maximum and minimum brightness.

(c) Graph the function b.



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

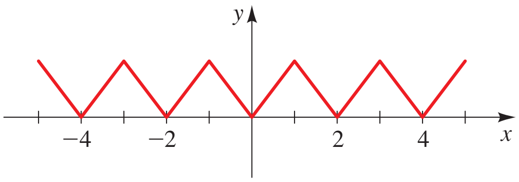
87. DISCUSS: Compositions Involving Trigonometric Functions This exercise explores the effect of the inner function g on a composite function $y = f(g(x))$.

(a) Graph the function $y = \sin\sqrt{x}$ using the viewing rectangle $[0, 400]$ by $[-1.5, 1.5]$. In what ways does this graph differ from the graph of the sine function?

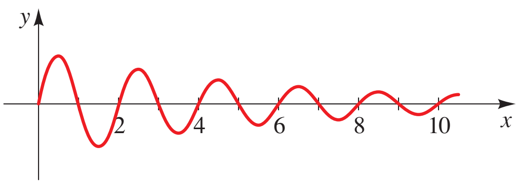
(b) Graph the function $y = \sin(x^2)$ using the viewing rectangle $[-5, 5]$ by $[-1.5, 1.5]$. In what ways does this graph differ from the graph of the sine function?

88. DISCUSS: Periodic Functions I Recall that a function f is *periodic* if there is a positive number p such that $f(t + p) = f(t)$ for every t, and the least such p (if it exists) is the *period* of f. The graph of a function of period p looks the same on each interval of length p, so we can easily determine the period from the graph. Determine whether the function whose graph is shown is periodic; if it is periodic, find the period.

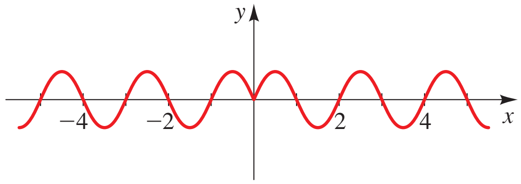
(a)



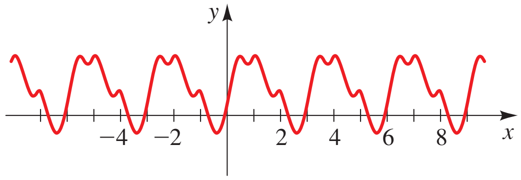
(b)



(c)



(d)



89. DISCUSS: Periodic Functions II Use a graphing device to graph the following functions. From the graph, determine whether the function is periodic; if it is periodic, find the period. (See page 163 for the definition of $[\![x]\!]$.)

(a) $y = |\sin x|$

(b) $y = \sin|x|$

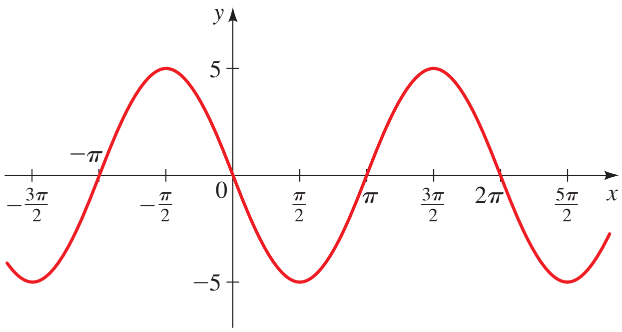
(c) $y = 2^{\cos x}$

(d) $y = x - [\![x]\!]$

(e) $y = \cos(\sin x)$

(f) $y = \cos(x^2)$

90. DISCUSS: Sinusoidal Curves The graph of $y = \sin x$ is the same as the graph of $y = \cos x$ shifted to the right $\pi/2$ units. So the sine curve $y = \sin x$ is also at the same time a cosine curve: $y = \cos(x - \frac{\pi}{2})$. In fact, any sine curve is also a cosine curve with a different horizontal shift, and any cosine curve is also a sine curve. Sine and cosine curves are collectively referred to as *sinusoidal*. For the curve whose graph is shown, find all possible ways of expressing it as a sine curve $y = a\sin(x - b)$ or as a cosine curve $y = a\cos(x - b)$. Explain why you think you have found all possible choices for a and b in each case.



5.4 MORE TRIGONOMETRIC GRAPHS

■ Graphs of Tangent, Cotangent, Secant, and Cosecant ■ Graphs of Transformations of Tangent and Cotangent ■ Graphs of Transformations of Cosecant and Secant

In this section we graph the tangent, cotangent, secant, and cosecant functions and transformations of these functions.

■ Graphs of Tangent, Cotangent, Secant, and Cosecant

We begin by stating the periodic properties of these functions. Recall that sine and cosine have period 2π. Since cosecant and secant are the reciprocals of sine and cosine, respectively, they also have period 2π (see Exercise 63). Tangent and cotangent, however, have period π (see Exercise 83 of Section 5.2).

PERIODIC PROPERTIES

The functions tangent and cotangent have period π:

$$\tan(x + \pi) = \tan x \qquad \cot(x + \pi) = \cot x$$

The functions cosecant and secant have period 2π:

$$\csc(x + 2\pi) = \csc x \qquad \sec(x + 2\pi) = \sec x$$

x	$\tan x$
0	0
$\pi/6$	0.58
$\pi/4$	1.00
$\pi/3$	1.73
1.4	5.80
1.5	14.10
1.55	48.08
1.57	1,255.77
1.5707	10,381.33

We first sketch the graph of tangent. Since it has period π, we need only sketch the graph on any interval of length π and then repeat the pattern to the left and to the right. We sketch the graph on the interval $(-\pi/2, \pi/2)$. Since $\tan(\pi/2)$ and $\tan(-\pi/2)$ aren't defined, we need to be careful in sketching the graph at points near $\pi/2$ and $-\pi/2$. As x gets near $\pi/2$ through values less than $\pi/2$, the value of $\tan x$ becomes large. To see this, notice that as x gets close to $\pi/2$, $\cos x$ approaches 0 and $\sin x$ approaches 1 and so $\tan x = \sin x/\cos x$ is large. A table of values of $\tan x$ for x close to $\pi/2$ (≈ 1.570796) is shown in the margin.

So as x approaches $\pi/2$ from the left, the value of $\tan x$ increases without bound. We express this by writing

$$\tan x \to \infty \quad \text{as} \quad x \to \frac{\pi}{2}^{-}$$

This is read "tan x approaches infinity as x approaches $\pi/2$ from the left."

In a similar way, as x approaches $-\pi/2$ from the right, the value of $\tan x$ decreases without bound. We write this as

Arrow notation is discussed in Section 3.6.

$$\tan x \to -\infty \quad \text{as} \quad x \to -\frac{\pi}{2}^{+}$$

This is read "tan x approaches negative infinity as x approaches $-\pi/2$ from the right."

Asymptotes are discussed in Section 3.6.

Thus the graph of $y = \tan x$ approaches the vertical lines $x = \pi/2$ and $x = -\pi/2$. So these lines are **vertical asymptotes**. With the information we have so far, we sketch the graph of $y = \tan x$ for $-\pi/2 < x < \pi/2$ in Figure 1. The complete graph of tangent (see

Mathematics in the Modern World

Evaluating Functions on a Calculator

How does your calculator evaluate sin t, cos t, e^t, ln t, $\sqrt{t}$, and other such functions? One method is to approximate these functions by polynomials because polynomials are easy to evaluate. For example,

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. These remarkable formulas were found by the British mathematician Brook Taylor (1685–1731). For instance, if we use the first three terms of Taylor's series to find cos(0.4), we get

$$\cos 0.4 \approx 1 - \frac{(0.4)^2}{2!} + \frac{(0.4)^4}{4!}$$

$$\approx 0.92106667$$

(Compare this with the value you get from your calculator.) The graph shows that the more terms of the series we use, the more closely the polynomials approximate the function cos t.

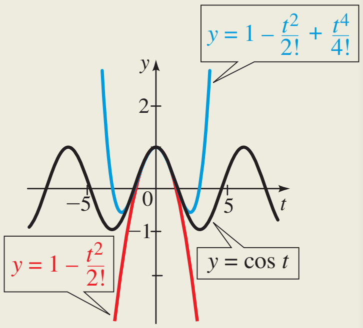


Figure 5(a) on the next page) is now obtained by using the fact that tangent is periodic with period π.

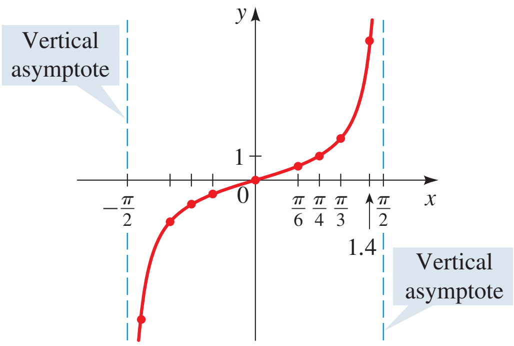


FIGURE 1 One period of $y = \tan x$

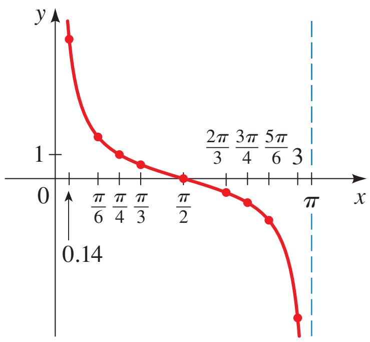


FIGURE 2 One period of $y = \cot x$

The function $y = \cot x$ is graphed on the interval $(0, \pi)$ by a similar analysis (see Figure 2). Since $\cot x$ is undefined for $x = n\pi$ with n an integer, its complete graph (in Figure 5(b) on the next page) has vertical asymptotes at these values.

To graph the cosecant and secant functions, we use the reciprocal identities

$$\csc x = \frac{1}{\sin x} \qquad \text{and} \qquad \sec x = \frac{1}{\cos x}$$

So to graph $y = \csc x$, we take the reciprocals of the y-coordinates of the points of the graph of $y = \sin x$. (See Figure 3.) Similarly, to graph $y = \sec x$, we take the reciprocals of the y-coordinates of the points of the graph of $y = \cos x$. (See Figure 4.)

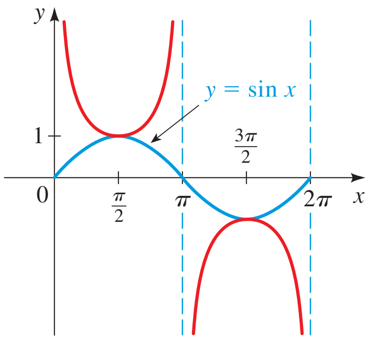


FIGURE 3 One period of $y = \csc x$

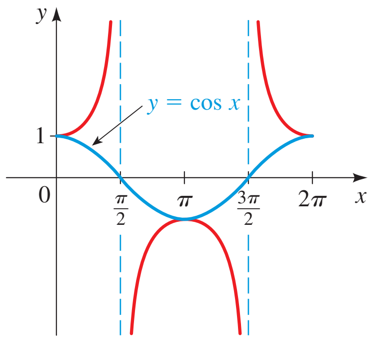


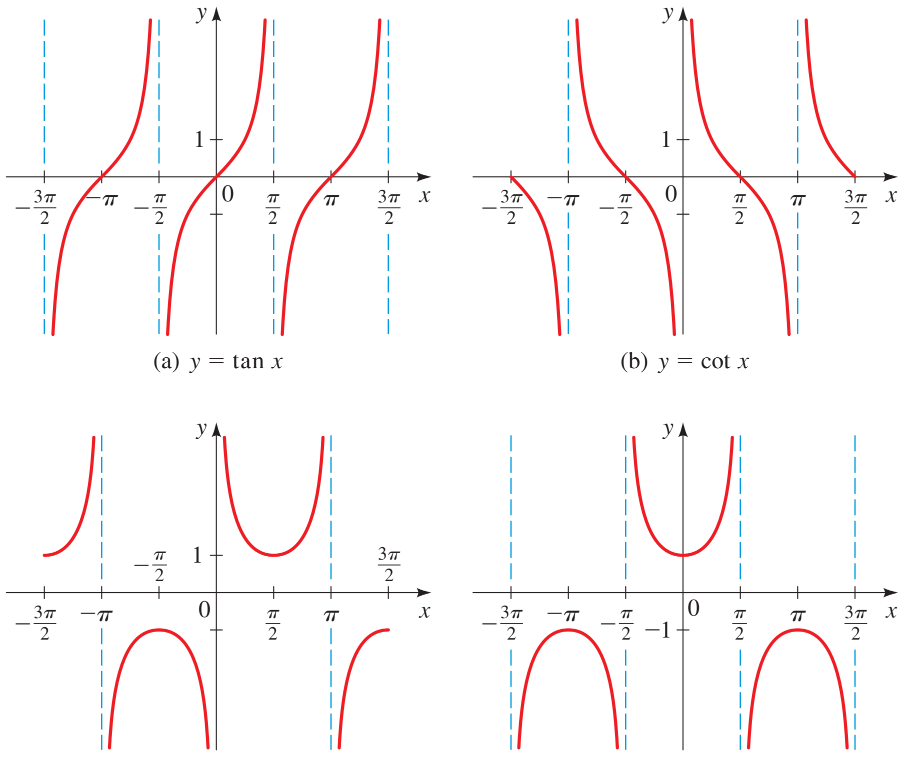
FIGURE 4 One period of $y = \sec x$

Let's consider more closely the graph of the function $y = \csc x$ on the interval $0 < x < \pi$. We need to examine the values of the function near 0 and π, since at these values $\sin x = 0$, and $\csc x$ is thus undefined. We see that

$$\csc x \to \infty \qquad \text{as} \qquad x \to 0^+$$

$$\csc x \to \infty \qquad \text{as} \qquad x \to \pi^-$$

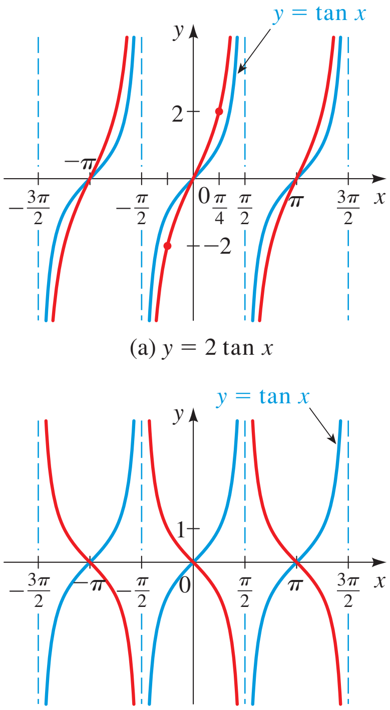
Thus the lines $x = 0$ and $x = \pi$ are vertical asymptotes. In the interval $\pi < x < 2\pi$ the graph is sketched in the same way. The values of $\csc x$ in that interval are the same as those in the interval $0 < x < \pi$ except for sign (see Figure 3). The complete graph in Figure 5(c) is now obtained from the fact that the function cosecant is periodic with period 2π. Note that the graph has vertical asymptotes at the points where $\sin x = 0$, that is, at $x = n\pi$, for n an integer.



(a) $y = \tan x$ (b) $y = \cot x$

(c) $y = \csc x$ (d) $y = \sec x$

FIGURE 5



(a) $y = 2 \tan x$

(b) $y = -\tan x$

FIGURE 6

The graph of $y = \sec x$ is sketched in a similar manner. Observe that the domain of $\sec x$ is the set of all real numbers other than $x = (\pi/2) + n\pi$, for n an integer, so the graph has vertical asymptotes at those points. The complete graph is shown in Figure 5(d).

It is apparent that the graphs of $y = \tan x$, $y = \cot x$, and $y = \csc x$ are symmetric about the origin, whereas that of $y = \sec x$ is symmetric about the y-axis. This is because tangent, cotangent, and cosecant are odd functions, whereas secant is an even function.

Graphs of Transformations of Tangent and Cotangent

We now consider graphs of transformations of the tangent and cotangent functions.

EXAMPLE 1 ■ Graphing Tangent Curves

Graph each function.

(a) $y = 2 \tan x$ **(b)** $y = -\tan x$

SOLUTION We first graph $y = \tan x$ and then transform it as required.

(a) To graph $y = 2 \tan x$, we multiply the y-coordinate of each point on the graph of $y = \tan x$ by 2. The resulting graph is shown in Figure 6(a).

(b) The graph of $y = -\tan x$ in Figure 6(b) is obtained from that of $y = \tan x$ by reflecting in the x-axis.

Now Try Exercises 9 and 11

Since the tangent and cotangent functions have period π, the functions

$$y = a \tan kx \qquad \text{and} \qquad y = a \cot kx \qquad (k > 0)$$

complete one period as kx varies from 0 to π, that is, for $0 \le kx \le \pi$. Solving this inequality, we get $0 \le x \le \pi/k$. So they each have period π/k.

TANGENT AND COTANGENT CURVES

The functions

$$y = a \tan kx \qquad \text{and} \qquad y = a \cot kx \qquad (k > 0)$$

have period π/k.

Thus one complete period of the graphs of these functions occurs on any interval of length π/k. To sketch a complete period of these graphs, it's convenient to select an interval between vertical asymptotes:

To graph one period of $y = a \tan kx$, an appropriate interval is $\left(-\dfrac{\pi}{2k}, \dfrac{\pi}{2k}\right)$.

To graph one period of $y = a \cot kx$, an appropriate interval is $\left(0, \dfrac{\pi}{k}\right)$.

EXAMPLE 2 ■ Graphing Tangent Curves

Graph each function.

(a) $y = \tan 2x$ **(b)** $y = \tan 2\left(x - \dfrac{\pi}{4}\right)$

SOLUTION

(a) The period is $\pi/2$ and an appropriate interval is $(-\pi/4, \pi/4)$. The endpoints $x = -\pi/4$ and $x = \pi/4$ are vertical asymptotes. Thus we graph one complete period of the function on $(-\pi/4, \pi/4)$. The graph has the same shape as that of the tangent function but is shrunk horizontally by a factor of $\frac{1}{2}$. We then repeat that portion of the graph to the left and to the right. See Figure 7(a).

(b) The graph is the same as that in part (a), but it is shifted to the right $\pi/4$, as shown in Figure 7(b).

Since $y = \tan x$ completes one period between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, the function $y = \tan 2(x - \frac{\pi}{4})$ completes one period as $2(x - \frac{\pi}{4})$ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Start of period:	End of period:
$2(x - \frac{\pi}{4}) = -\frac{\pi}{2}$	$2(x - \frac{\pi}{4}) = \frac{\pi}{2}$
$x - \frac{\pi}{4} = -\frac{\pi}{4}$	$x - \frac{\pi}{4} = \frac{\pi}{4}$
$x = 0$	$x = \frac{\pi}{2}$

So we graph one period on the interval $(0, \frac{\pi}{2})$.

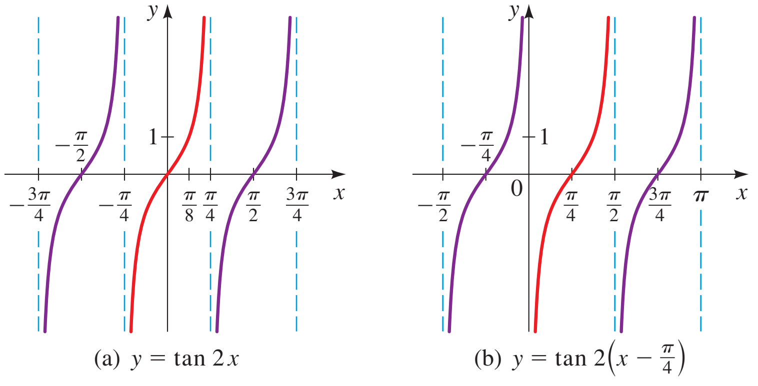


FIGURE 7

Now Try Exercises 19, 35, and 43 ■

EXAMPLE 3 ■ A Horizontally Shifted Cotangent Curve

Graph the function $y = 2\cot\left(3x - \dfrac{\pi}{4}\right)$.

SOLUTION We first put the equation in the form $y = a \cot k(x - b)$ by factoring 3 from the expression $3x - \dfrac{\pi}{4}$:

$$y = 2\cot\left(3x - \frac{\pi}{4}\right) = 2\cot 3\left(x - \frac{\pi}{12}\right)$$

Thus the graph is the same as that of $y = 2\cot 3x$ but is shifted to the right $\pi/12$. The period of $y = 2\cot 3x$ is $\pi/3$, and an appropriate interval for graphing one period is $(0, \pi/3)$. To get the corresponding interval for the desired graph, we shift this interval to the right $\pi/12$. So we have

$$\left(0 + \frac{\pi}{12}, \frac{\pi}{3} + \frac{\pi}{12}\right) = \left(\frac{\pi}{12}, \frac{5\pi}{12}\right)$$

Finally, we graph one period in the shape of cotangent on the interval $(\pi/12, 5\pi/12)$ and repeat that portion of the graph to the left and to the right. (See Figure 8.)

Since $y = \cot x$ completes one period between $x = 0$ and $x = \pi$, the function $y = 2\cot\left(3x - \frac{\pi}{4}\right)$ completes one period as $3x - \frac{\pi}{4}$ varies from 0 to π.

Start of period:	End of period:
$3x - \frac{\pi}{4} = 0$	$3x - \frac{\pi}{4} = \pi$
$3x = \frac{\pi}{4}$	$3x = \frac{5\pi}{4}$
$x = \frac{\pi}{12}$	$x = \frac{5\pi}{12}$

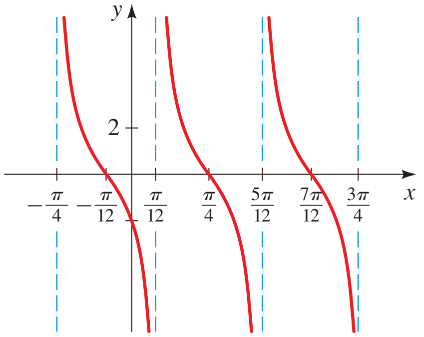


FIGURE 8
$y = 2\cot\left(3x - \dfrac{\pi}{4}\right)$

Now Try Exercises 37 and 47 ■

■ Graphs of Transformations of Cosecant and Secant

We have already observed that the cosecant and secant functions are the reciprocals of the sine and cosine functions. Thus the following result is the counterpart of the result for sine and cosine curves in Section 5.3.

COSECANT AND SECANT CURVES

The functions

$$y = a \csc kx \quad \text{and} \quad y = a \sec kx \quad (k > 0)$$

have period $2\pi/k$.

An appropriate interval on which to graph one complete period is $(0, 2\pi/k)$.

EXAMPLE 4 ■ Graphing Cosecant Curves

Graph each function.

(a) $y = \dfrac{1}{2}\csc 2x$ **(b)** $y = \dfrac{1}{2}\csc\left(2x + \dfrac{\pi}{2}\right)$

SOLUTION

(a) The period is $2\pi/2 = \pi$. An appropriate interval is $[0, \pi]$, and the asymptotes occur in this interval whenever $\sin 2x = 0$. So the asymptotes in this interval are $x = 0$, $x = \pi/2$, and $x = \pi$. With this information we sketch on the interval $[0, \pi]$ a graph with the same general shape as that of one period of the cosecant function. The complete graph in Figure 9(a) is obtained by repeating this portion of the graph to the left and to the right.

(b) We first write

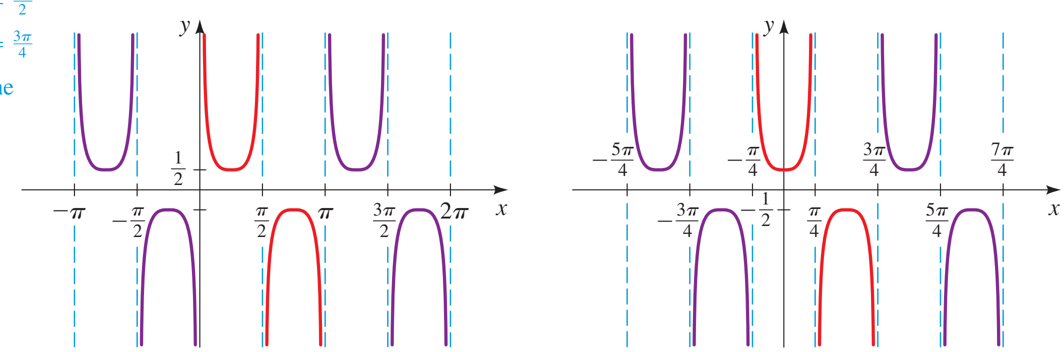
$$y = \frac{1}{2}\csc\left(2x + \frac{\pi}{2}\right) = \frac{1}{2}\csc 2\left(x + \frac{\pi}{4}\right)$$

From this we see that the graph is the same as that in part (a) but shifted to the left $\pi/4$. The graph is shown in Figure 9(b).

Since $y = \csc x$ completes one period between $x = 0$ and $x = 2\pi$, the function $y = \frac{1}{2}\csc(2x + \frac{\pi}{2})$ completes one period as $2x + \frac{\pi}{2}$ varies from 0 to 2π.

Start of period:	End of period:
$2x + \frac{\pi}{2} = 0$	$2x + \frac{\pi}{2} = 2\pi$
$2x = -\frac{\pi}{2}$	$2x = \frac{3\pi}{2}$
$x = -\frac{\pi}{4}$	$x = \frac{3\pi}{4}$

So we graph one period on the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$.



(a) $y = \frac{1}{2}\csc 2x$ (b) $y = \frac{1}{2}\csc\left(2x + \frac{\pi}{2}\right)$

FIGURE 9

Now Try Exercises 29 and 49 ■

EXAMPLE 5 ■ Graphing a Secant Curve

Graph $y = 3\sec\frac{1}{2}x$.

SOLUTION The period is $2\pi \div \frac{1}{2} = 4\pi$. An appropriate interval is $[0, 4\pi]$, and the asymptotes occur in this interval wherever $\cos\frac{1}{2}x = 0$. Thus the asymptotes in this interval are $x = \pi$, $x = 3\pi$. With this information we sketch on the interval $[0, 4\pi]$ a graph with the same general shape as that of one period of the secant function. The complete graph in Figure 10 is obtained by repeating this portion of the graph to the left and to the right.

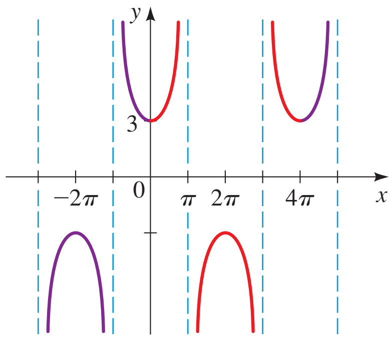


FIGURE 10
$y = 3\sec\frac{1}{2}x$

Now Try Exercises 31 and 51 ■

5.4 EXERCISES

CONCEPTS

1. The trigonometric function $y = \tan x$ has period ________ and asymptotes $x =$ ________. Sketch a graph of this function on the interval $(-\pi/2, \pi/2)$.

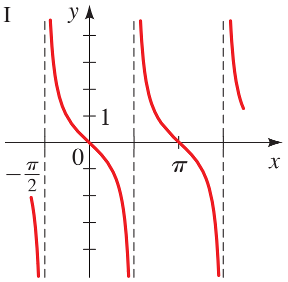
2. The trigonometric function $y = \csc x$ has period ________ and asymptotes $x =$ ________. Sketch a graph of this function on the interval $(-\pi, \pi)$.

SKILLS

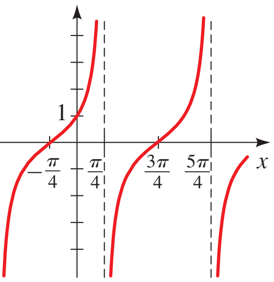
3–8 ■ Graphs of Trigonometric Functions Match the trigonometric function with one of the graphs I–VI.

3. $f(x) = \tan\left(x + \dfrac{\pi}{4}\right)$
4. $f(x) = \sec 2x$
5. $f(x) = \cot 4x$
6. $f(x) = -\tan x$
7. $f(x) = 2 \sec x$
8. $f(x) = 1 + \csc x$

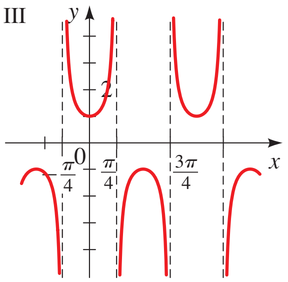
I



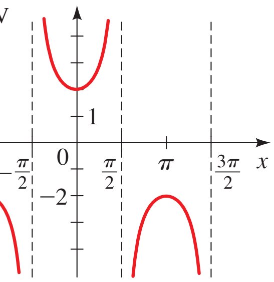
II



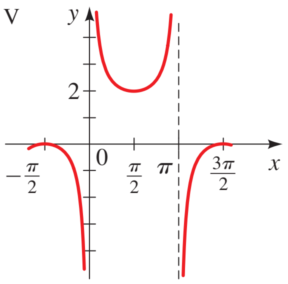
III



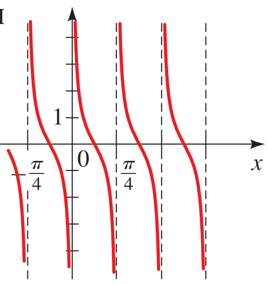
IV



V



VI



9–18 ■ Graphs of Trigonometric Functions Find the period, and graph the function.

9. $y = 3 \tan x$
10. $y = -3 \tan x$
11. $y = -\frac{3}{2} \tan x$
12. $y = \frac{3}{4} \tan x$
13. $y = -\cot x$
14. $y = 2 \cot x$
15. $y = 2 \csc x$
16. $y = \frac{1}{2} \csc x$
17. $y = 3 \sec x$
18. $y = -3 \sec x$

19–34 ■ Graphs of Trigonometric Functions with Different Periods Find the period, and graph the function.

19. $y = \tan 3x$
20. $y = \tan 4x$
21. $y = -5 \tan \pi x$
22. $y = -3 \tan 4\pi x$
23. $y = 2 \cot 3\pi x$
24. $y = 3 \cot 2\pi x$
25. $y = \tan \dfrac{\pi}{4} x$
26. $y = \cot \dfrac{\pi}{2} x$
27. $y = 2 \tan 3\pi x$
28. $y = 2 \tan \dfrac{\pi}{2} x$
29. $y = \csc 4x$
30. $y = 5 \csc 3x$
31. $y = \sec 2x$
32. $y = \frac{1}{2} \sec(4\pi x)$
33. $y = 5 \csc \dfrac{3\pi}{2} x$
34. $y = 5 \sec 2\pi x$

35–60 ■ Graphs of Trigonometric Functions with Horizontal Shifts Find the period, and graph the function.

35. $y = \tan\left(x + \dfrac{\pi}{4}\right)$
36. $y = \tan\left(x - \dfrac{\pi}{4}\right)$
37. $y = \cot\left(x + \dfrac{\pi}{4}\right)$
38. $y = 2 \cot\left(x - \dfrac{\pi}{3}\right)$
39. $y = \csc\left(x - \dfrac{\pi}{4}\right)$
40. $y = \sec\left(x + \dfrac{\pi}{4}\right)$
41. $y = \dfrac{1}{2} \sec\left(x - \dfrac{\pi}{6}\right)$
42. $y = 3 \csc\left(x + \dfrac{\pi}{2}\right)$
43. $y = \tan 2\left(x - \dfrac{\pi}{3}\right)$
44. $y = \cot\left(2x - \dfrac{\pi}{4}\right)$
45. $y = 5 \cot\left(3x + \dfrac{\pi}{2}\right)$
46. $y = 4 \tan(4x - 2\pi)$
47. $y = \cot\left(2x - \dfrac{\pi}{2}\right)$
48. $y = \frac{1}{2} \tan(\pi x - \pi)$
49. $y = 2 \csc\left(\pi x - \dfrac{\pi}{3}\right)$
50. $y = 3 \sec\left(\dfrac{1}{4} x - \dfrac{\pi}{6}\right)$
51. $y = \sec 2\left(x - \dfrac{\pi}{4}\right)$
52. $y = \csc 2\left(x + \dfrac{\pi}{2}\right)$
53. $y = 5 \sec\left(3x - \dfrac{\pi}{2}\right)$
54. $y = \frac{1}{2} \sec(2\pi x - \pi)$
55. $y = \tan\left(\dfrac{2}{3} x - \dfrac{\pi}{6}\right)$
56. $y = \tan \dfrac{1}{2}\left(x + \dfrac{\pi}{4}\right)$

57. $y = 3 \sec \pi\left(x + \frac{1}{2}\right)$

58. $y = \sec\left(3x + \frac{\pi}{2}\right)$

59. $y = -2 \tan\left(2x - \frac{\pi}{3}\right)$

60. $y = 2 \cot(3\pi x + 3\pi)$

APPLICATIONS

61. Lighthouse The beam from a lighthouse completes one rotation every 2 min. At time t, the distance d shown in the figure below is

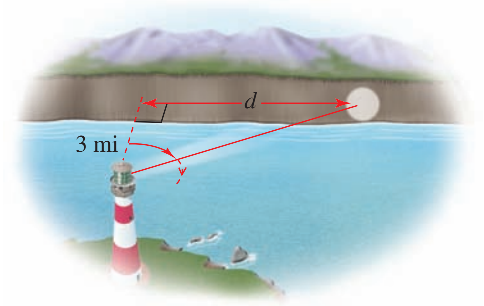
$$d(t) = 3 \tan \pi t$$

where t is measured in minutes and d in miles.

(a) Find $d(0.15)$, $d(0.25)$, and $d(0.45)$.

(b) Sketch a graph of the function d for $0 \le t < \frac{1}{2}$.

(c) What happens to the distance d as t approaches $\frac{1}{2}$?



62. Length of a Shadow On a day when the sun passes directly overhead at noon, a 6-ft-tall man casts a shadow of length

$$S(t) = 6\left|\cot \frac{\pi}{12} t\right|$$

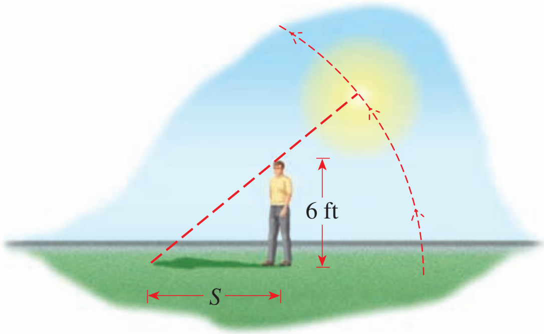
where S is measured in feet and t is the number of hours since 6 A.M.

(a) Find the length of the shadow at 8:00 A.M., noon, 2:00 P.M., and 5:45 P.M.

(b) Sketch a graph of the function S for $0 < t < 12$.

(c) From the graph, determine the values of t at which the length of the shadow equals the man's height. To what time of day does each of these values correspond?

(d) Explain what happens to the shadow as the time approaches 6 P.M. (that is, as $t \to 12^-$).



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

63. PROVE: Periodic Functions **(a)** Prove that if f is periodic with period p, then $1/f$ is also periodic with period p.

(b) Prove that cosecant and secant both have period 2π.

64. PROVE: Periodic Functions Prove that if f and g are periodic with period p, then f/g is also periodic but its period could be smaller than p.

65. PROVE: Reduction Formulas Use the graphs in Figure 5 to explain why the following formulas are true.

$$\tan\left(x - \frac{\pi}{2}\right) = -\cot x \qquad \sec\left(x - \frac{\pi}{2}\right) = \csc x$$

5.5 INVERSE TRIGONOMETRIC FUNCTIONS AND THEIR GRAPHS

■ The Inverse Sine Function ■ The Inverse Cosine Function ■ The Inverse Tangent Function ■ The Inverse Secant, Cosecant, and Cotangent Functions

We study applications of inverse trigonometric functions to triangles in Sections 6.4–6.6.

Recall from Section 2.8 that the inverse of a function f is a function f^{-1} that reverses the rule of f. For a function to have an inverse, it must be one-to-one. Since the trigonometric functions are not one-to-one, they do not have inverses. It is possible, however, to restrict the domains of the trigonometric functions in such a way that the resulting functions are one-to-one.

■ The Inverse Sine Function

Let's first consider the sine function. There are many ways to restrict the domain of sine so that the new function is one-to-one. A natural way to do this is to restrict the domain to the interval $[-\pi/2, \pi/2]$. The reason for this choice is that sine is one-to-one on this interval and moreover attains each of the values in its range on this interval. From Figure 1 we see that sine is one-to-one on this restricted domain (by the Horizontal Line Test) and so has an inverse.

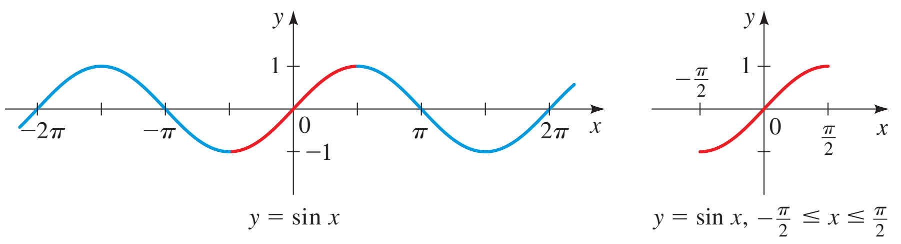


FIGURE 1 Graphs of the sine function and the restricted sine function

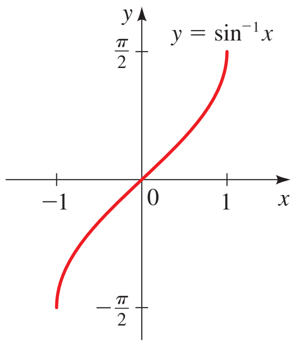


FIGURE 2 Graph of $y = \sin^{-1} x$

We can now define an inverse sine function on this restricted domain. The graph of $y = \sin^{-1} x$ is shown in Figure 2; it is obtained by reflecting the graph of $y = \sin x$, $-\pi/2 \le x \le \pi/2$, in the line $y = x$.

DEFINITION OF THE INVERSE SINE FUNCTION

The **inverse sine function** is the function $\sin^{-1}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

The inverse sine function is also called **arcsine**, denoted by **arcsin**.

Thus $y = \sin^{-1} x$ *is the number in the interval* $[-\pi/2, \pi/2]$ *whose sine is* x. In other words, $\sin(\sin^{-1} x) = x$. In fact, from the general properties of inverse functions studied in Section 2.8, we have the following **cancellation properties**.

$$\sin(\sin^{-1} x) = x \quad \text{for} \quad -1 \le x \le 1$$

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

EXAMPLE 1 ■ Evaluating the Inverse Sine Function

Find each value.

(a) $\sin^{-1} \frac{1}{2}$ **(b)** $\sin^{-1}\left(-\frac{1}{2}\right)$ **(c)** $\sin^{-1} \frac{3}{2}$

SOLUTION

(a) The number in the interval $[-\pi/2, \pi/2]$ whose sine is $\frac{1}{2}$ is $\pi/6$. Thus $\sin^{-1} \frac{1}{2} = \pi/6$.

(b) The number in the interval $[-\pi/2, \pi/2]$ whose sine is $-\frac{1}{2}$ is $-\pi/6$. Thus $\sin^{-1}(-\frac{1}{2}) = -\pi/6$.

(c) Since $\frac{3}{2} > 1$, it is not in the domain of $\sin^{-1} x$, so $\sin^{-1} \frac{3}{2}$ is not defined.

Now Try Exercise 3 ■

EXAMPLE 2 ■ Using a Calculator to Evaluate Inverse Sine

Find approximate values for **(a)** $\sin^{-1}(0.82)$ and **(b)** $\sin^{-1}\frac{1}{3}$.

SOLUTION

We use a calculator to approximate these values. Using the [SIN⁻¹], or [INV] [SIN], or [ARC] [SIN] key(s) on the calculator (with the calculator in radian mode), we get

(a) $\sin^{-1}(0.82) \approx 0.96141$ **(b)** $\sin^{-1}\frac{1}{3} \approx 0.33984$

Now Try Exercises 11 and 21 ■

When evaluating expressions involving $\sin^{-1}$, we need to remember that the range of $\sin^{-1}$ is the interval $[-\pi/2, \pi/2]$.

EXAMPLE 3 ■ Evaluating Expressions with Inverse Sine

Find each value.

(a) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ **(b)** $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

SOLUTION

(a) Since $\pi/3$ is in the interval $[-\pi/2, \pi/2]$, we can use the cancellation properties of inverse functions (page 440):

$$\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3} \qquad \text{Cancellation property: } -\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}$$

(b) We first evaluate the expression in the parentheses:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \qquad \text{Evaluate}$$

$$= \frac{\pi}{3} \qquad \text{Because } \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Note: $\sin^{-1}(\sin x) = x$ only if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Now Try Exercises 31 and 33 ■

■ The Inverse Cosine Function

If the domain of the cosine function is restricted to the interval $[0, \pi]$, the resulting function is one-to-one and so has an inverse. We choose this interval because on it, cosine attains each of its values exactly once (see Figure 3).

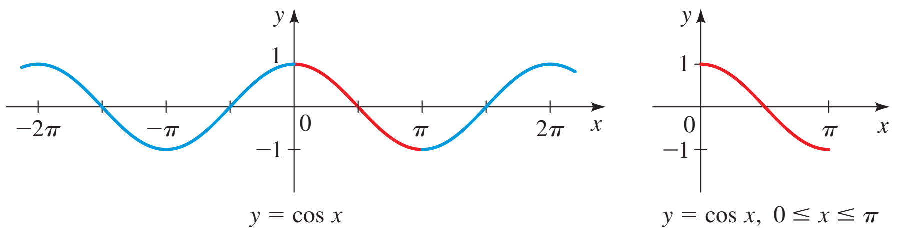


FIGURE 3 Graphs of the cosine function and the restricted cosine function

DEFINITION OF THE INVERSE COSINE FUNCTION

The **inverse cosine function** is the function $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

The inverse cosine function is also called **arccosine**, denoted by **arccos**.

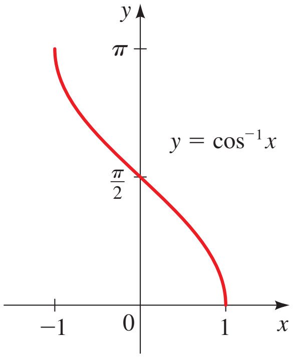


FIGURE 4 Graph of $y = \cos^{-1} x$

Thus $y = \cos^{-1} x$ *is the number in the interval* $[0, \pi]$ *whose cosine is* x. The following **cancellation properties** follow from the inverse function properties.

$$\cos(\cos^{-1} x) = x \quad \text{for} \quad -1 \le x \le 1$$
$$\cos^{-1}(\cos x) = x \quad \text{for} \quad 0 \le x \le \pi$$

The graph of $y = \cos^{-1} x$ is shown in Figure 4; it is obtained by reflecting the graph of $y = \cos x$, $0 \le x \le \pi$, in the line $y = x$.

EXAMPLE 4 ■ Evaluating the Inverse Cosine Function

Find each value.

(a) $\cos^{-1} \dfrac{\sqrt{3}}{2}$ **(b)** $\cos^{-1} 0$ **(c)** $\cos^{-1}\left(-\dfrac{1}{2}\right)$

SOLUTION

(a) The number in the interval $[0, \pi]$ whose cosine is $\sqrt{3}/2$ is $\pi/6$. Thus $\cos^{-1}(\sqrt{3}/2) = \pi/6$.

(b) The number in the interval $[0, \pi]$ whose cosine is 0 is $\pi/2$. Thus $\cos^{-1} 0 = \pi/2$.

(c) The number in the interval $[0, \pi]$ whose cosine is $-\frac{1}{2}$ is $2\pi/3$. Thus $\cos^{-1}(-\frac{1}{2}) = 2\pi/3$. (The graph in Figure 4 shows that if $-1 \le x < 0$, then $\cos^{-1} x > \pi/2$.)

Now Try Exercises 5 and 13 ■

EXAMPLE 5 ■ Evaluating Expressions with Inverse Cosine

Find each value.

(a) $\cos^{-1}\left(\cos \dfrac{2\pi}{3}\right)$ **(b)** $\cos^{-1}\left(\cos \dfrac{5\pi}{3}\right)$

SOLUTION

(a) Since $2\pi/3$ is in the interval $[0, \pi]$ we can use the above cancellation properties:

$$\cos^{-1}\left(\cos \frac{2\pi}{3}\right) = \frac{2\pi}{3} \qquad \text{Cancellation property: } 0 \le \frac{2\pi}{3} \le \pi$$

(b) We first evaluate the expression in the parentheses:

$$\cos^{-1}\left(\cos \frac{5\pi}{3}\right) = \cos^{-1}\left(\frac{1}{2}\right) \qquad \text{Evaluate}$$
$$= \frac{\pi}{3} \qquad \text{Because } \cos \frac{\pi}{3} = \frac{1}{2}$$

Note: $\cos^{-1}(\cos x) = x$ only if $0 \le x \le \pi$.

Now Try Exercises 35 and 37 ■

■ The Inverse Tangent Function

We restrict the domain of the tangent function to the interval $(-\pi/2, \pi/2)$ to obtain a one-to-one function.

DEFINITION OF THE INVERSE TANGENT FUNCTION

The **inverse tangent function** is the function $\tan^{-1}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

The inverse tangent function is also called **arctangent**, denoted by **arctan**.

Thus $y = \tan^{-1} x$ *is the number in the interval* $(-\pi/2, \pi/2)$ *whose tangent is* x. The following **cancellation properties** follow from the inverse function properties.

$$\tan(\tan^{-1} x) = x \quad \text{for} \quad x \in \mathbb{R}$$

$$\tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Figure 5 shows the graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$ and the graph of its inverse function, $y = \tan^{-1} x$.

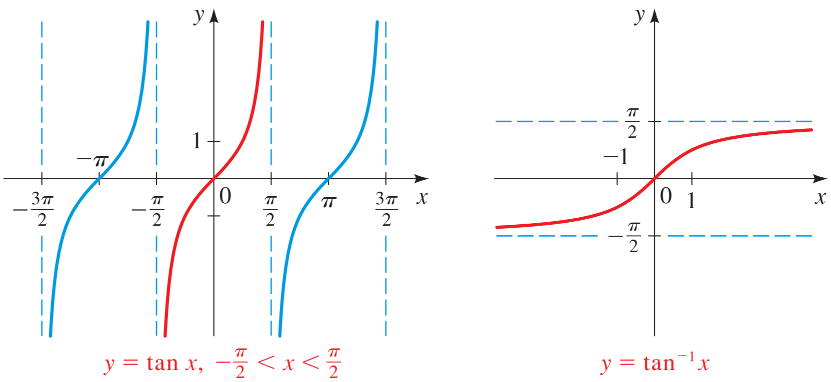


FIGURE 5 Graphs of the restricted tangent function and the inverse tangent function

EXAMPLE 6 ■ Evaluating the Inverse Tangent Function

Find each value.

(a) $\tan^{-1} 1$ **(b)** $\tan^{-1} \sqrt{3}$ **(c)** $\tan^{-1}(20)$

SOLUTION

(a) The number in the interval $(-\pi/2, \pi/2)$ with tangent 1 is $\pi/4$. Thus $\tan^{-1} 1 = \pi/4$.

(b) The number in the interval $(-\pi/2, \pi/2)$ with tangent $\sqrt{3}$ is $\pi/3$. Thus $\tan^{-1} \sqrt{3} = \pi/3$.

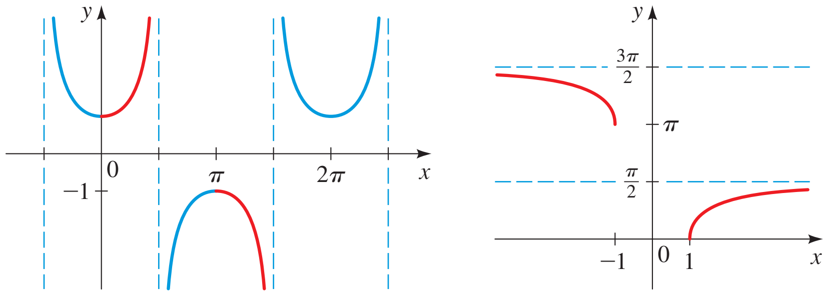
(c) We use a calculator (in radian mode) to find that $\tan^{-1}(20) \approx 1.52084$.

Now Try Exercises 7 and 17 ■

■ The Inverse Secant, Cosecant, and Cotangent Functions

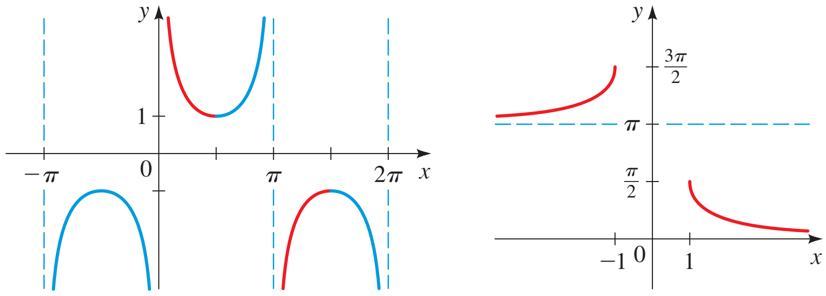
See Exercise 46 in Section 6.4 (page 508) for a way of finding the values of these inverse trigonometric functions on a calculator.

To define the inverse functions of the secant, cosecant, and cotangent functions, we restrict the domain of each function to a set on which it is one-to-one and on which it attains all its values. Although any interval satisfying these criteria is appropriate, we choose to restrict the domains in a way that simplifies the choice of sign in computations involving inverse trigonometric functions. The choices we make are also appropriate for calculus. This explains the seemingly strange restriction for the domains of the secant and cosecant functions. We end this section by displaying the graphs of the secant, cosecant, and cotangent functions with their restricted domains and the graphs of their inverse functions (Figures 6–8).



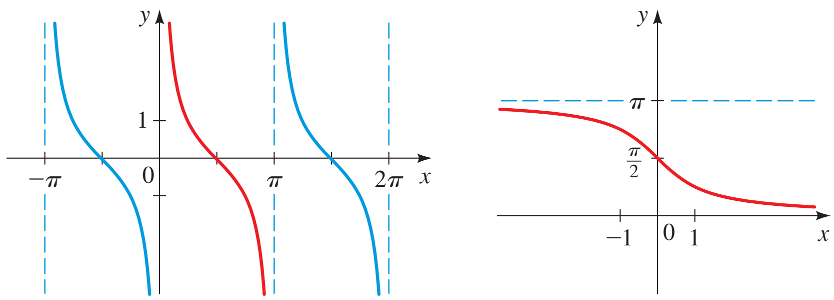
$y = \sec x,\ 0 \le x < \frac{\pi}{2},\ \pi \le x < \frac{3\pi}{2}$ $\qquad y = \sec^{-1} x$

FIGURE 6 The inverse secant function



$y = \csc x,\ 0 < x \le \frac{\pi}{2},\ \pi < x \le \frac{3\pi}{2}$ $\qquad y = \csc^{-1} x$

FIGURE 7 The inverse cosecant function



$y = \cot x,\ 0 < x < \pi$ $\qquad y = \cot^{-1} x$

FIGURE 8 The inverse cotangent function

5.5 EXERCISES

CONCEPTS

1. (a) To define the inverse sine function, we restrict the domain of sine to the interval ____________. On this interval the sine function is one-to-one, and its inverse function $\sin^{-1}$ is defined by $\sin^{-1} x = y \Leftrightarrow \sin$ ______ = ______. For example, $\sin^{-1} \frac{1}{2} =$ ______ because sin ______ = ______.

(b) To define the inverse cosine function, we restrict the domain of cosine to the interval ____________. On this interval the cosine function is one-to-one and its inverse function $\cos^{-1}$ is defined by $\cos^{-1} x = y \Leftrightarrow$ cos ______ = ______. For example, $\cos^{-1} \frac{1}{2} =$ ______ because cos ______ = ______.

2. The cancellation property $\sin^{-1}(\sin x) = x$ is valid for x in the interval ____________. Which of the following is not true?

(i) $\sin^{-1}\left(\sin \frac{\pi}{3}\right) = \frac{\pi}{3}$ $\qquad$ (ii) $\sin^{-1}\left(\sin \frac{10\pi}{3}\right) = \frac{10\pi}{3}$

(iii) $\sin^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

SKILLS

3–10 ■ Evaluating Inverse Trigonometric Functions Find the exact value of each expression, if it is defined.

3. (a) $\sin^{-1} 1$ $\qquad$ **(b)** $\sin^{-1} \frac{\sqrt{3}}{2}$ $\qquad$ **(c)** $\sin^{-1} 2$

4. **(a)** $\sin^{-1}(-1)$ **(b)** $\sin^{-1}\dfrac{\sqrt{2}}{2}$ **(c)** $\sin^{-1}(-2)$

5. **(a)** $\cos^{-1}(-1)$ **(b)** $\cos^{-1}\frac{1}{2}$ **(c)** $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

6. **(a)** $\cos^{-1}\dfrac{\sqrt{2}}{2}$ **(b)** $\cos^{-1} 1$ **(c)** $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

7. **(a)** $\tan^{-1}(-1)$ **(b)** $\tan^{-1}\sqrt{3}$ **(c)** $\tan^{-1}\dfrac{\sqrt{3}}{3}$

8. **(a)** $\tan^{-1} 0$ **(b)** $\tan^{-1}(-\sqrt{3})$ **(c)** $\tan^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)$

9. **(a)** $\cos^{-1}\left(-\frac{1}{2}\right)$ **(b)** $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$ **(c)** $\tan^{-1} 1$

10. **(a)** $\cos^{-1} 0$ **(b)** $\sin^{-1} 0$ **(c)** $\sin^{-1}\left(-\frac{1}{2}\right)$

11–22 ■ Inverse Trigonometric Functions with a Calculator Use a calculator to find an approximate value of each expression correct to five decimal places, if it is defined.

11. $\sin^{-1}\frac{2}{3}$ **12.** $\sin^{-1}\left(-\frac{8}{9}\right)$

13. $\cos^{-1}\left(-\frac{3}{7}\right)$ **14.** $\cos^{-1}\left(\frac{4}{9}\right)$

15. $\cos^{-1}(-0.92761)$ **16.** $\sin^{-1}(0.13844)$

17. $\tan^{-1} 10$ **18.** $\tan^{-1}(-26)$

19. $\tan^{-1}(1.23456)$ **20.** $\cos^{-1}(1.23456)$

21. $\sin^{-1}(-0.25713)$ **22.** $\tan^{-1}(-0.25713)$

23–48 ■ Simplifying Expressions Involving Trigonometric Functions Find the exact value of the expression, if it is defined.

23. $\sin\left(\sin^{-1}\frac{1}{4}\right)$ **24.** $\cos\left(\cos^{-1}\frac{2}{3}\right)$

25. $\tan(\tan^{-1} 5)$ **26.** $\sin(\sin^{-1} 5)$

27. $\sin\left(\sin^{-1}\frac{3}{2}\right)$ **28.** $\tan\left(\tan^{-1}\frac{3}{2}\right)$

29. $\cos\left(\cos^{-1}\left(-\dfrac{1}{5}\right)\right)$ **30.** $\sin\left(\sin^{-1}\left(-\dfrac{3}{4}\right)\right)$

31. $\sin^{-1}\left(\sin\left(\dfrac{\pi}{4}\right)\right)$ **32.** $\cos^{-1}\left(\cos\left(\dfrac{\pi}{4}\right)\right)$

33. $\sin^{-1}\left(\sin\left(\dfrac{3\pi}{4}\right)\right)$ **34.** $\cos^{-1}\left(\cos\left(\dfrac{3\pi}{4}\right)\right)$

35. $\cos^{-1}\left(\cos\left(\dfrac{5\pi}{6}\right)\right)$ **36.** $\sin^{-1}\left(\sin\left(\dfrac{5\pi}{6}\right)\right)$

37. $\cos^{-1}\left(\cos\left(\dfrac{7\pi}{6}\right)\right)$ **38.** $\sin^{-1}\left(\sin\left(\dfrac{7\pi}{6}\right)\right)$

39. $\tan^{-1}\left(\tan\left(\dfrac{\pi}{4}\right)\right)$ **40.** $\tan^{-1}\left(\tan\left(-\dfrac{\pi}{3}\right)\right)$

41. $\tan^{-1}\left(\tan\left(\dfrac{2\pi}{3}\right)\right)$ **42.** $\sin^{-1}\left(\sin\left(\dfrac{11\pi}{4}\right)\right)$

43. $\tan\left(\sin^{-1}\frac{1}{2}\right)$ **44.** $\cos(\sin^{-1} 0)$

45. $\cos\left(\sin^{-1}\dfrac{\sqrt{3}}{2}\right)$ **46.** $\tan\left(\sin^{-1}\dfrac{\sqrt{2}}{2}\right)$

47. $\sin(\tan^{-1}(-1))$ **48.** $\sin(\tan^{-1}(-\sqrt{3}))$

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

49–50 ■ PROVE: Identities Involving Inverse Trigonometric Functions **(a)** Graph the function and make a conjecture, and **(b)** prove that your conjecture is true.

49. $y = \sin^{-1} x + \cos^{-1} x$ **50.** $y = \tan^{-1} x + \tan^{-1}\dfrac{1}{x}$

51. DISCUSS: Two Different Compositions Let f and g be the functions

$$f(x) = \sin(\sin^{-1} x)$$

and

$$g(x) = \sin^{-1}(\sin x)$$

By the cancellation properties, $f(x) = x$ and $g(x) = x$ for suitable values of x. But these functions are not the same for all x. Graph both f and g to show how the functions differ. (Think carefully about the domain and range of $\sin^{-1}$).

5.6 MODELING HARMONIC MOTION

■ Simple Harmonic Motion ■ Damped Harmonic Motion ■ Phase and Phase Difference

Periodic behavior—behavior that repeats over and over again—is common in nature. Perhaps the most familiar example is the daily rising and setting of the sun, which results in the repetitive pattern of day, night, day, night, ... Another example is the daily variation of tide levels at the beach, which results in the repetitive pattern of high tide, low tide, high tide, low tide, ... Certain animal populations increase and decrease in a predictable periodic pattern: A large population exhausts the food supply, which causes the population to dwindle; this in turn results in a more plentiful food supply, which makes it possible for the population to increase; and the pattern then repeats over and over (see *Discovery Project: Predator/Prey Models* referenced on page 427).

Other common examples of periodic behavior involve motion that is caused by vibration or oscillation. A mass suspended from a spring that has been compressed and then allowed to vibrate vertically is a simple example. This back-and-forth motion also occurs in such diverse phenomena as sound waves, light waves, alternating electrical current, and pulsating stars, to name a few. In this section we consider the problem of modeling periodic behavior.

Simple Harmonic Motion

The trigonometric functions are ideally suited for modeling periodic behavior. A glance at the graphs of the sine and cosine functions, for instance, tells us that these functions themselves exhibit periodic behavior. Figure 1 shows the graph of $y = \sin t$. If we think of t as time, we see that as time goes on, $y = \sin t$ increases and decreases over and over again. Figure 2 shows that the motion of a vibrating mass on a spring is modeled very accurately by $y = \sin t$.

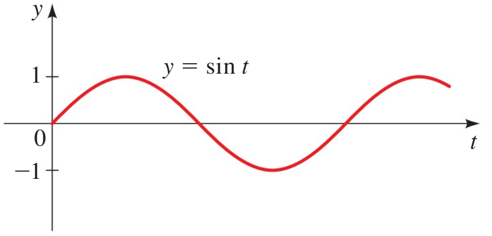


FIGURE 1 $y = \sin t$

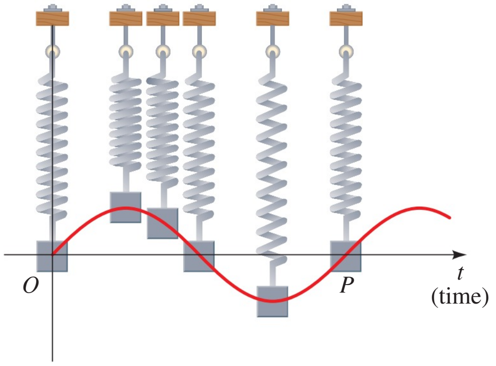


FIGURE 2 Motion of a vibrating spring is modeled by $y = \sin t$.

Notice that the mass returns to its original position over and over again. A **cycle** is one complete vibration of an object, so the mass in Figure 2 completes one cycle of its motion between O and P. Our observations about how the sine and cosine functions model periodic behavior are summarized in the following box.

The main difference between the two equations describing simple harmonic motion is the starting point. At $t = 0$ we get

$$y = a \sin \omega \cdot 0 = 0$$

$$y = a \cos \omega \cdot 0 = a$$

In the first case the motion "starts" with zero displacement, whereas in the second case the motion "starts" with the displacement at maximum (at the amplitude a).

The symbol ω is the lowercase Greek letter "omega," and ν is the letter "nu."

SIMPLE HARMONIC MOTION

If the equation describing the displacement y of an object at time t is

$$y = a \sin \omega t \qquad \text{or} \qquad y = a \cos \omega t$$

then the object is in **simple harmonic motion**. In this case,

amplitude $= |a|$ — Maximum displacement of the object

period $= \dfrac{2\pi}{\omega}$ — Time required to complete one cycle

frequency $= \dfrac{\omega}{2\pi}$ — Number of cycles per unit of time

Notice that the functions

$$y = a \sin 2\pi\nu t \qquad \text{and} \qquad y = a \cos 2\pi\nu t$$

have frequency ν, because $2\pi\nu/(2\pi) = \nu$. Since we can immediately read the frequency from these equations, we often write equations of simple harmonic motion in this form.

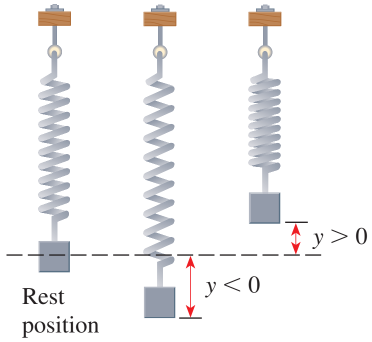


FIGURE 3

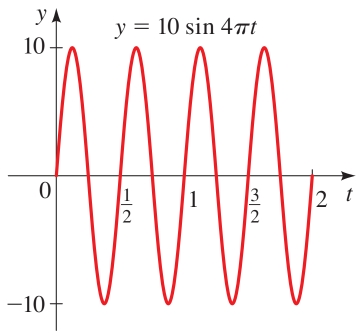


FIGURE 4

EXAMPLE 1 ■ A Vibrating Spring

The displacement of a mass suspended by a spring is modeled by the function

$$y = 10 \sin 4\pi t$$

where y is measured in inches and t in seconds (see Figure 3).

(a) Find the amplitude, period, and frequency of the motion of the mass.

(b) Sketch a graph of the displacement of the mass.

SOLUTION

(a) From the formulas for amplitude, period, and frequency we get

$$\text{amplitude} = |a| = 10 \text{ in.}$$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi} = \frac{1}{2} \text{ s}$$

$$\text{frequency} = \frac{\omega}{2\pi} = \frac{4\pi}{2\pi} = 2 \text{ cycles per second (Hz)}$$

(b) The graph of the displacement of the mass at time t is shown in Figure 4.

Now Try Exercise 5 ■

An important situation in which simple harmonic motion occurs is in the production of sound. Sound is produced by a regular variation in air pressure from the normal pressure. If the pressure varies in simple harmonic motion, then a pure sound is produced. The tone of the sound depends on the frequency, and the loudness depends on the amplitude.

EXAMPLE 2 ■ Vibrations of a Musical Note

A sousaphone player plays the note E and sustains the sound for some time. For a pure E the variation in pressure from normal air pressure is given by

$$V(t) = 0.2 \sin 80\pi t$$

where V is measured in pounds per square inch and t is measured in seconds.

(a) Find the amplitude, period, and frequency of V.

(b) Sketch a graph of V.

(c) If the player increases the loudness of the note, how does the equation for V change?

(d) If the player is playing the note incorrectly and it is a little flat, how does the equation for V change?

SOLUTION

(a) From the formulas for amplitude, period, and frequency we get

$$\text{amplitude} = |0.2| = 0.2$$

$$\text{period} = \frac{2\pi}{80\pi} = \frac{1}{40}$$

$$\text{frequency} = \frac{80\pi}{2\pi} = 40$$

(b) The graph of V is shown in Figure 5.

(c) If the player increases the loudness the amplitude increases. So the number 0.2 is replaced by a larger number.

(d) If the note is flat, then the frequency is decreased. Thus the coefficient of t is less than 80π.

Now Try Exercise 41 ■

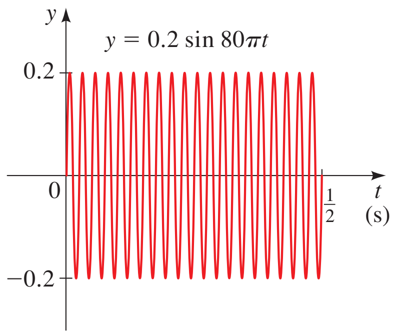


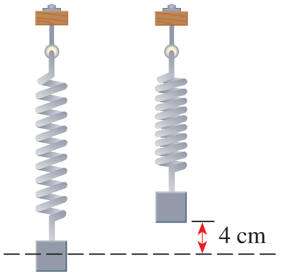
FIGURE 5

EXAMPLE 3 ■ Modeling a Vibrating Spring

A mass is suspended from a spring. The spring is compressed a distance of 4 cm and then released. It is observed that the mass returns to the compressed position after $\frac{1}{3}$ s.

(a) Find a function that models the displacement of the mass.

(b) Sketch the graph of the displacement of the mass.



SOLUTION

(a) The motion of the mass is given by one of the equations for simple harmonic motion. The amplitude of the motion is 4 cm. Since this amplitude is reached at time $t = 0$, an appropriate function that models the displacement is of the form

$$y = a \cos \omega t$$

Since the period is $p = \frac{1}{3}$, we can find ω from the following equation:

$$\text{period} = \frac{2\pi}{\omega}$$

$$\frac{1}{3} = \frac{2\pi}{\omega} \qquad \text{Period} = \tfrac{1}{3}$$

$$\omega = 6\pi \qquad \text{Solve for } \omega$$

So the motion of the mass is modeled by the function

$$y = 4 \cos 6\pi t$$

where y is the displacement from the rest position at time t. Notice that when $t = 0$, the displacement is $y = 4$, as we expect.

(b) The graph of the displacement of the mass at time t is shown in Figure 6.

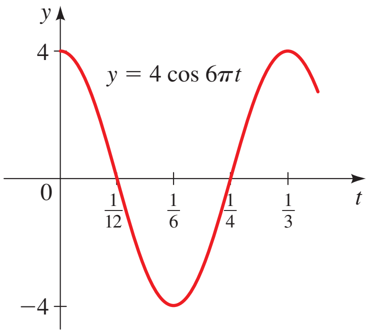


FIGURE 6

Now Try Exercises 17 and 47 ■

In general, the sine or cosine functions representing harmonic motion may be shifted horizontally or vertically. In this case the equations take the form

$$y = a \sin(\omega(t - c)) + b \qquad \text{or} \qquad y = a \cos(\omega(t - c)) + b$$

The vertical shift b indicates that the variation occurs around an average value b. The horizontal shift c indicates the position of the object at $t = 0$. (See Figure 7.)

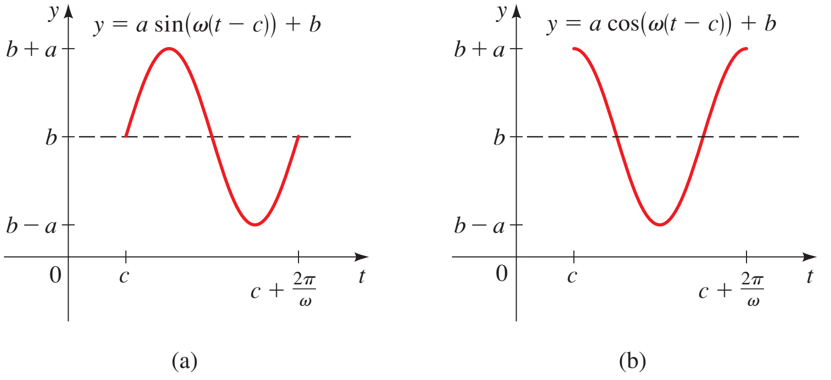


FIGURE 7

EXAMPLE 4 ■ Modeling the Brightness of a Variable Star

A variable star is one whose brightness alternately increases and decreases. For the variable star Delta Cephei the time between periods of maximum brightness is 5.4 days. The average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude.

(a) Find a function that models the brightness of Delta Cephei as a function of time.

(b) Sketch a graph of the brightness of Delta Cephei as a function of time.

SOLUTION

(a) Let's find a function in the form

$$y = a\cos(\omega(t - c)) + b$$

The amplitude is the maximum variation from average brightness, so the amplitude is $a = 0.35$ magnitude. We are given that the period is 5.4 days, so

$$\omega = \frac{2\pi}{5.4} \approx 1.16$$

Since the brightness varies from an average value of 4.0 magnitudes, the graph is shifted upward by $b = 4.0$. If we take $t = 0$ to be a time when the star is at maximum brightness, there is no horizontal shift, so $c = 0$ (because a cosine curve achieves its maximum at $t = 0$). Thus the function we want is

$$y = 0.35\cos(1.16t) + 4.0$$

where t is the number of days from a time when the star is at maximum brightness.

(b) The graph is sketched in Figure 8.

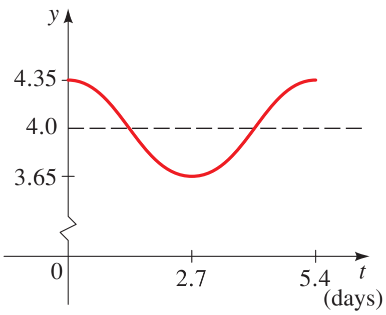


FIGURE 8

Now Try Exercise 51 ■

The number of hours of daylight varies throughout the course of a year. In the Northern Hemisphere the longest day is June 21, and the shortest is December 21. The average length of daylight is 12 h, and the variation from this average depends on the latitude. (For example, Fairbanks, Alaska, experiences more than 20 h of daylight on the longest day and less than 4 h on the shortest day!) The graph in Figure 9 shows the number of hours of daylight at different times of the year for various latitudes. It's apparent from the graph that the variation in hours of daylight is simple harmonic.

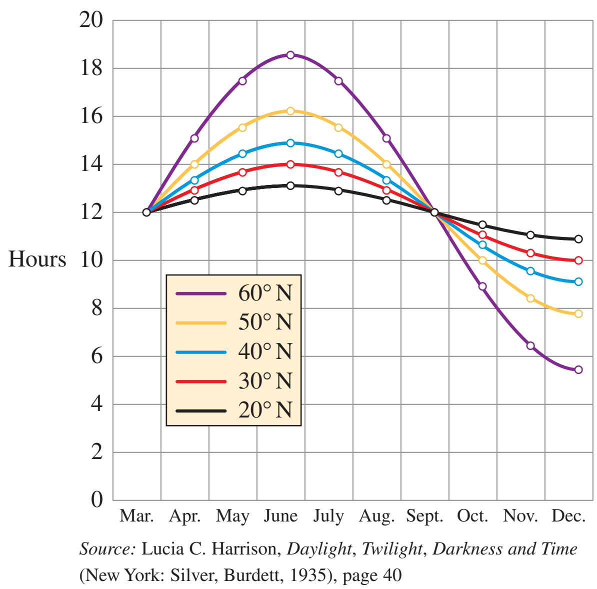


FIGURE 9 Graph of the length of daylight from March 21 through December 21 at various latitudes

EXAMPLE 5 ■ Modeling the Number of Hours of Daylight

In Philadelphia (40° N latitude) the longest day of the year has 14 h 50 min of daylight, and the shortest day has 9 h 10 min of daylight.

(a) Find a function L that models the length of daylight as a function of t, the number of days from January 1.

(b) An astronomer needs at least 11 hours of darkness for a long exposure astronomical photograph. On what days of the year are such long exposures possible?

SOLUTION

(a) We need to find a function in the form

$$y = a \sin(\omega(t - c)) + b$$

whose graph is the 40° N latitude curve in Figure 9. From the information given, we see that the amplitude is

$$a = \tfrac{1}{2}\left(14\tfrac{5}{6} - 9\tfrac{1}{6}\right) \approx 2.83 \text{ h}$$

Since there are 365 days in a year, the period is 365, so

$$\omega = \frac{2\pi}{365} \approx 0.0172$$

Since the average length of daylight is 12 h, the graph is shifted upward by 12, so $b = 12$. Since the curve attains the average value (12) on March 21, the 80th day of the year, the curve is shifted 80 units to the right. Thus $c = 80$. So a function that models the number of hours of daylight is

$$y = 2.83 \sin(0.0172(t - 80)) + 12$$

where t is the number of days from January 1.

(b) A day has 24 h, so 11 h of night correspond to 13 h of daylight. So we need to solve the inequality $y \le 13$. To solve this inequality graphically, we graph $y = 2.83 \sin 0.0172(t - 80) + 12$ and $y = 13$ on the same graph. From the graph in Figure 10 we see that there are fewer than 13 h of daylight between day 1 (January 1) and day 101 (April 11) and between day 241 (August 29) and day 365 (December 31).

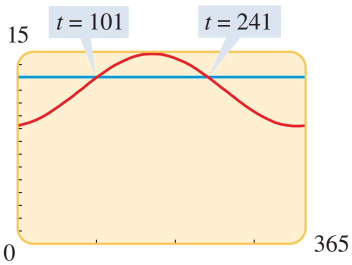


FIGURE 10

Now Try Exercise 53 ■

Another situation in which simple harmonic motion occurs is in alternating current (AC) generators. Alternating current is produced when an armature rotates about its axis in a magnetic field.

Figure 11 represents a simple version of such a generator. As the wire passes through the magnetic field, a voltage E is generated in the wire. It can be shown that the voltage generated is given by

$$E(t) = E_0 \cos \omega t$$

where E_0 is the maximum voltage produced (which depends on the strength of the magnetic field) and $\omega/(2\pi)$ is the number of revolutions per second of the armature (the frequency).

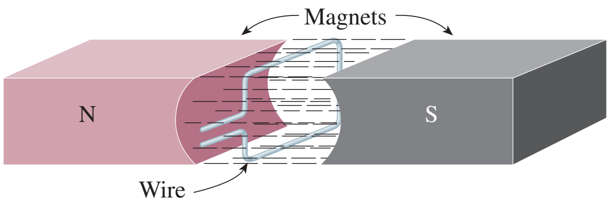


FIGURE 11

Why do we say that household current is 110 V when the maximum voltage produced is 155 V? From the symmetry of the cosine function we see that the average voltage produced is zero. This average value would be the same for all AC generators and so gives no information about the voltage generated. To obtain a more informative measure of voltage, engineers use the **root-mean-square** (RMS) method. It can be shown that the RMS voltage is $1/\sqrt{2}$ times the maximum voltage. So for household current the RMS voltage is

$$155 \times \frac{1}{\sqrt{2}} \approx 110 \text{ V}$$

EXAMPLE 6 ■ Modeling Alternating Current

Ordinary 110-V household alternating current varies from $+155$ V to -155 V with a frequency of 60 Hz (cycles per second). Find an equation that describes this variation in voltage.

SOLUTION The variation in voltage is simple harmonic. Since the frequency is 60 cycles per second, we have

$$\frac{\omega}{2\pi} = 60 \qquad \text{or} \qquad \omega = 120\pi$$

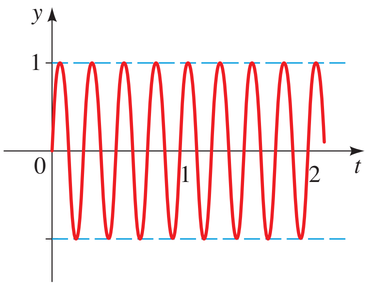
Let's take $t = 0$ to be a time when the voltage is $+155$ V. Then

$$E(t) = a\cos\omega t = 155\cos 120\pi t$$

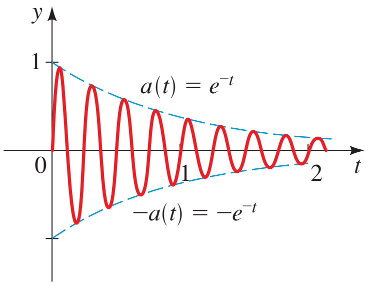
Now Try Exercise 55 ■

Damped Harmonic Motion

The spring in Figure 2 on page 446 is assumed to oscillate in a frictionless environment. In this hypothetical case the amplitude of the oscillation will not change. In the presence of friction, however, the motion of the spring eventually "dies down"; that is, the amplitude of the motion decreases with time. Motion of this type is called *damped harmonic motion*.



(a) Harmonic motion: $y = \sin 8\pi t$



(b) Damped harmonic motion: $y = e^{-t}\sin 8\pi t$

FIGURE 12

DAMPED HARMONIC MOTION

If the equation describing the displacement y of an object at time t is

$$y = ke^{-ct}\sin\omega t \qquad \text{or} \qquad y = ke^{-ct}\cos\omega t \qquad (c > 0)$$

then the object is in **damped harmonic motion**. The constant c is the **damping constant**, k is the initial amplitude, and $2\pi/\omega$ is the period.*

Damped harmonic motion is simply harmonic motion for which the amplitude is governed by the function $a(t) = ke^{-ct}$. Figure 12 shows the difference between harmonic motion and damped harmonic motion.

EXAMPLE 7 ■ Modeling Damped Harmonic Motion

Two mass-spring systems are experiencing damped harmonic motion, both at 0.5 cycles per second and both with an initial maximum displacement of 10 cm. The first has a damping constant of 0.5, and the second has a damping constant of 0.1.

(a) Find functions of the form $g(t) = ke^{-ct}\cos\omega t$ to model the motion in each case.

(b) Graph the two functions you found in part (a). How do they differ?

Hz is the abbreviation for hertz. One hertz is one cycle per second.

SOLUTION

(a) At time $t = 0$ the displacement is 10 cm. Thus $g(0) = ke^{-c\cdot 0}\cos(\omega \cdot 0) = k$, so $k = 10$. Also, the frequency is $f = 0.5$ Hz, and since $\omega = 2\pi f$ (see page 446), we get $\omega = 2\pi(0.5) = \pi$. Using the given damping constants, we find that the motions of the two springs are given by the functions

$$g_1(t) = 10e^{-0.5t}\cos\pi t \qquad \text{and} \qquad g_2(t) = 10e^{-0.1t}\cos\pi t$$

*In the case of damped harmonic motion the term *quasi-period* is often used instead of *period* because the motion is not actually periodic—it diminishes with time. However, we will continue to use the term *period* to avoid confusion.

(b) The functions g_1 and g_2 are graphed in Figure 13. From the graphs we see that in the first case (where the damping constant is larger) the motion dies down quickly, whereas in the second case, perceptible motion continues much longer.

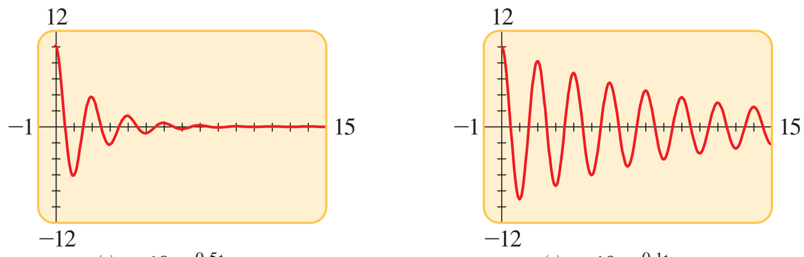


FIGURE 13 $g_1(t) = 10\, e^{-0.5t} \cos \pi t$ $g_2(t) = 10\, e^{-0.1t} \cos \pi t$

Now Try Exercise 21 ■

As Example 7 indicates, the larger the damping constant c, the quicker the oscillation dies down. When a guitar string is plucked and then allowed to vibrate freely, a point on that string undergoes damped harmonic motion. We hear the damping of the motion as the sound produced by the vibration of the string fades. How fast the damping of the string occurs (as measured by the size of the constant c) is a property of the size of the string and the material it is made of. Another example of damped harmonic motion is the motion that a shock absorber on a car undergoes when the car hits a bump in the road. In this case the shock absorber is engineered to damp the motion as quickly as possible (large c) and to have the frequency as small as possible (small ω). On the other hand, the sound produced by a tuba player playing a note is undamped as long as the player can maintain the loudness of the note. The electromagnetic waves that produce light move in simple harmonic motion that is not damped.

EXAMPLE 8 ■ A Vibrating Violin String

The G-string on a violin is pulled a distance of 0.5 cm above its rest position, then released and allowed to vibrate. The damping constant c for this string is determined to be 1.4. Suppose that the note produced is a pure G (frequency $= 200$ Hz). Find an equation that describes the motion of the point at which the string was plucked.

SOLUTION Let P be the point at which the string was plucked. We will find a function $f(t)$ that gives the distance at time t of the point P from its original rest position. Since the maximum displacement occurs at $t = 0$, we find an equation in the form

$$y = ke^{-ct} \cos \omega t$$

From this equation we see that $f(0) = k$. But we know that the original displacement of the string is 0.5 cm. Thus $k = 0.5$. Since the frequency of the vibration is 200, we have $\omega = 2\pi f = 2\pi(200) = 400\pi$. Finally, since we know that the damping constant is 1.4, we get

$$f(t) = 0.5e^{-1.4t} \cos 400\pi t$$

Now Try Exercise 57 ■

EXAMPLE 9 ■ Ripples on a Pond

A stone is dropped in a calm lake, causing waves to form. The up-and-down motion of a point on the surface of the water is modeled by damped harmonic motion. At some time the amplitude of the wave is measured, and 20 s later it is found that the amplitude has dropped to $\frac{1}{10}$ of this value. Find the damping constant c.

SOLUTION The amplitude is governed by the coefficient ke^{-ct} in the equations for damped harmonic motion. Thus the amplitude at time t is ke^{-ct}, and 20 s later, it is $ke^{-c(t+20)}$. So because the later value is $\frac{1}{10}$ the earlier value, we have

$$ke^{-c(t+20)} = \tfrac{1}{10}ke^{-ct}$$

We now solve this equation for c. Canceling k and using the Laws of Exponents, we get

$$e^{-ct} \cdot e^{-20c} = \tfrac{1}{10}e^{-ct}$$

$$e^{-20c} = \tfrac{1}{10} \qquad \text{Cancel } e^{-ct}$$

$$e^{20c} = 10 \qquad \text{Take reciprocals}$$

Taking the natural logarithm of each side gives

$$20c = \ln(10)$$

$$c = \tfrac{1}{20}\ln(10) \approx \tfrac{1}{20}(2.30) \approx 0.12$$

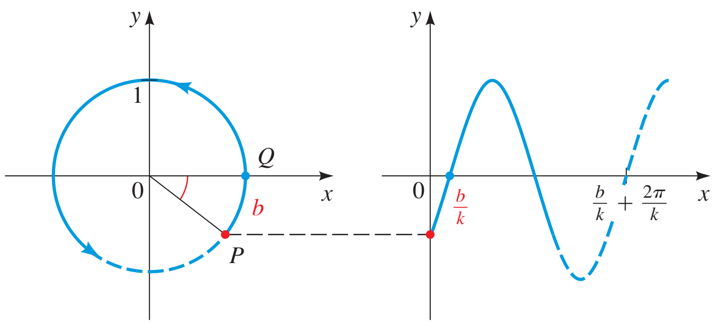
Thus the damping constant is $c \approx 0.12$.

Now Try Exercise 59 ■

Phase and Phase Difference

When two objects are moving in simple harmonic motion with the same frequency, it is often important to determine whether the objects are "moving together" or by how much their motions differ. Let's consider a specific example.

Suppose that an object is rotating along the unit circle and the height y of the object at time t is given by $y = \sin(kt - b)$. When $t = 0$, the height is $y = \sin(-b)$. This means that the motion "starts" at an angle b as shown in Figure 14.



(a) Phase angle b (b) Horizontal shift $\frac{b}{k}$

FIGURE 14 Graph of $y = \sin(kt - b)$

The phase angle b depends only on the starting position of the object and not on the frequency. The lag time does depend on the frequency.

We can view the starting point in two ways: as the *angle* between P and Q on the unit circle or as the *time* required for P to "catch up" to Q. The angle b is called the **phase** (or **phase angle**). To find the time required, we factor out k:

$$y = \sin(kt - b) = \sin k\left(t - \frac{b}{k}\right)$$

We see that P "catches up" to Q (that is, $y = 0$) when $t = b/k$. This last equation also shows that the graph in Figure 14(b) is **shifted horizontally** b/k (to the right) on the t-axis. The time b/k is called the **lag time** if $b > 0$ (because P is behind, or lags, Q by b/k time units) and is called the **lead time** if $b < 0$.

PHASE

Any sine curve can be expressed in the following equivalent forms:

$$y = A\sin(kt - b) \qquad \text{The } \textbf{phase} \text{ is } b.$$

$$y = A\sin k\left(t - \frac{b}{k}\right) \qquad \text{The } \textbf{horizontal shift} \text{ is } \frac{b}{k}.$$

It is often important to know whether two waves with the same period (modeled by sine curves) are *in phase* or *out of phase*. For the curves

$$y_1 = A\sin(kt - b) \qquad \text{and} \qquad y_2 = A\sin(kt - c)$$

the **phase difference** between y_1 and y_2 is $b - c$. If the phase difference is a multiple of 2π, the waves are **in phase**; otherwise, the waves are **out of phase**. If two sine curves are in phase, then their graphs coincide.

Note that the phase difference depends on the order in which the functions are given.

EXAMPLE 10 ■ Finding Phase and Phase Difference

Objects are in harmonic motion modeled by the following curves:

$$y_1 = 10\sin\left(3t - \frac{\pi}{6}\right) \qquad y_2 = 10\sin\left(3t - \frac{\pi}{2}\right) \qquad y_3 = 10\sin\left(3t + \frac{23\pi}{6}\right)$$

(a) Find the amplitude, period, phase, and horizontal shift of the curve y_1.

(b) Find the phase difference between the curves y_1 and y_2. Are the two curves in phase?

(c) Find the phase difference between the curves y_1 and y_3. Are the two curves in phase?

(d) Sketch all three curves on the same axes.

SOLUTION

(a) The amplitude is 10, the period is $2\pi/3$, and the phase is $\pi/6$. To find the horizontal shift, we factor:

$$y_1 = 10\sin\left(3t - \frac{\pi}{6}\right) = 10\sin 3\left(t - \frac{\pi}{18}\right)$$

So the horizontal shift is $\pi/18$.

(b) The phase of y_2 is $\pi/2$. So the phase difference is

$$\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

The phase difference is not a multiple of 2π, so the two curves are out of phase.

(c) The phase of y_3 is $-23\pi/6$. So the phase difference is

$$\frac{\pi}{6} - \left(-\frac{23\pi}{6}\right) = 4\pi = 2(2\pi)$$

The phase difference is a multiple of 2π, so the two curves are in phase.

(d) The graphs are shown in Figure 15. Notice that the curves y_1 and y_3 have the same graph because they are in phase.

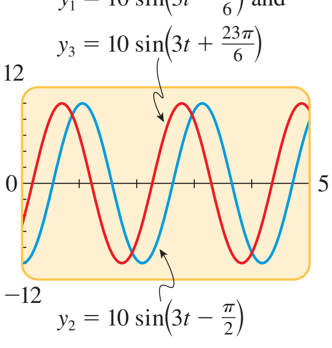
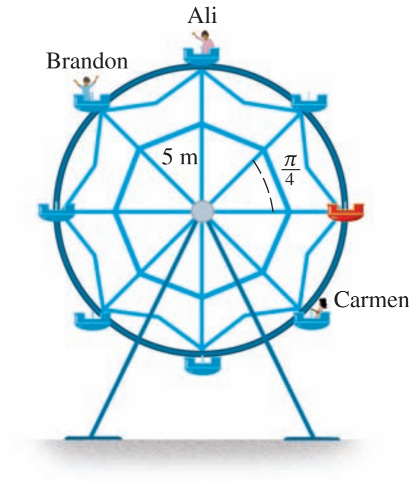


FIGURE 15

Now Try Exercises 29 and 35 ■



EXAMPLE 11 ■ Using Phase

Ali, Brandon, and Carmen are sitting in a stopped Ferris wheel as shown in the figure in the margin. At time $t = 0$ the Ferris wheel starts turning counterclockwise at the rate of 2 revolutions per minute.

(a) Find sine curves that model the height of each rider above the center line of the Ferris wheel at any time $t > 0$.

(b) Find the phase difference between Brandon and Ali, between Ali and Carmen, and between Brandon and Carmen.

(c) Find the horizontal shift of Ali's equation. What is Ali's lead or lag time (relative to the red seat in the figure)?

SOLUTION

(a) The motion of each rider is modeled by a function of the form $y = A\sin(kt - b)$. From the figure we see that the amplitude is $A = 5$ m. Since the Ferris wheel makes two revolutions per minute, the period is $\frac{1}{2}$ min. So

$$\text{period} = \frac{2\pi}{k} = \frac{1}{2}\text{ min}$$

It follows that $k = 4\pi$. From the figure we see that each rider starts at a different phase. Let's consider Ali and Brandon to be ahead of the red seat, and let's consider Carmen to be behind the red seat. So their phases are $-\pi/2$, $-3\pi/4$, and $\pi/4$, respectively. The equations are as follows.

Ali	Brandon	Carmen
$y_A = 5\sin\left(4\pi t + \frac{\pi}{2}\right)$	$y_B = 5\sin\left(4\pi t + \frac{3\pi}{4}\right)$	$y_C = 5\sin\left(4\pi t - \frac{\pi}{4}\right)$

(b) The phase differences are as follows.

Ali and Brandon	Ali and Carmen	Brandon and Carmen
$\frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$	$\frac{\pi}{2} - \left(-\frac{\pi}{4}\right) = \frac{3\pi}{4}$	$\frac{3\pi}{4} - \left(-\frac{\pi}{4}\right) = \pi$

(c) The equation that models Ali's position above the center line of the Ferris wheel was found in part (b). To find the horizontal shift, we factor Ali's equation.

$$y_A = 5\sin\left(4\pi t + \frac{\pi}{2}\right) \qquad \text{Ali's equation}$$

$$y_A = 5\sin 4\pi\left(t + \frac{1}{8}\right) \qquad \text{Factor } 4\pi$$

We see that the horizontal shift is $\frac{1}{8}$ to the left. This means that Ali's lead time is $\frac{1}{8}$ of a minute (so she is $\frac{1}{8}$ of a minute ahead of the red seat).

Now Try Exercise 61 ■

5.6 EXERCISES

CONCEPTS

1. For an object in simple harmonic motion with amplitude a and period $2\pi/\omega$, find an equation that models the displacement y at time t if
   (a) $y = 0$ at time $t = 0$: $y =$ ____________.
   (b) $y = a$ at time $t = 0$: $y =$ ____________.

2. For an object in damped harmonic motion with initial amplitude a, period $2\pi/\omega$, and damping constant c, find an equation that models the displacement y at time t if
   (a) $y = 0$ at time $t = 0$: $y =$ ____________.
   (b) $y = a$ at time $t = 0$: $y =$ ____________.

3. **(a)** For an object in harmonic motion modeled by $y = A\sin(kt - b)$ the amplitude is ____________, the period is ____________, and the phase is ____________. To find the horizontal shift, we factor out k to get $y =$ ____________. From this form of the equation we see that the horizontal shift is ____________.
   (b) For an object in harmonic motion modeled by $y = 5\sin(4t - \pi)$ the amplitude is ____________, the period is ____________, the phase is ____________, and the horizontal shift is ____________.

4. Objects A and B are in harmonic motion modeled by $y = 3\sin(2t - \pi)$ and $y = 3\sin\left(2t - \frac{\pi}{2}\right)$. The phase of A is ____________, and the phase of B is ____________. The phase difference is ____________, so the objects are moving ____________ (in phase/out of phase).

SKILLS

5–12 ■ Simple Harmonic Motion The given function models the displacement of an object moving in simple harmonic motion.
(a) Find the amplitude, period, and frequency of the motion.
(b) Sketch a graph of the displacement of the object over one complete period.

5. $y = 2\sin 3t$
6. $y = 3\cos \frac{1}{2}t$
7. $y = -\cos 0.3t$
8. $y = 2.4\sin 3.6t$
9. $y = -0.25\cos\left(1.5t - \frac{\pi}{3}\right)$
10. $y = -\frac{3}{2}\sin(0.2t + 1.4)$
11. $y = 5\cos\left(\frac{2}{3}t + \frac{3}{4}\right)$
12. $y = 1.6\sin(t - 1.8)$

13–16 ■ Simple Harmonic Motion Find a function that models the simple harmonic motion having the given properties. Assume that the displacement is zero at time $t = 0$.

13. amplitude 10 cm, period 3 s
14. amplitude 24 ft, period 2 min
15. amplitude 6 in., frequency $5/\pi$ Hz
16. amplitude 1.2 m, frequency 0.5 Hz

17–20 ■ Simple Harmonic Motion Find a function that models the simple harmonic motion having the given properties. Assume that the displacement is at its maximum at time $t = 0$.

17. amplitude 60 ft, period 0.5 min
18. amplitude 35 cm, period 8 s
19. amplitude 2.4 m, frequency 750 Hz
20. amplitude 6.25 in., frequency 60 Hz

21–28 ■ Damped Harmonic Motion An initial amplitude k, damping constant c, and frequency f or period p are given. (Recall that frequency and period are related by the equation $f = 1/p$.)
(a) Find a function that models the damped harmonic motion. Use a function of the form $y = ke^{-ct}\cos\omega t$ in Exercises 21–24 and of the form $y = ke^{-ct}\sin\omega t$ in Exercises 25–28.
(b) Graph the function.

21. $k = 2$, $c = 1.5$, $f = 3$
22. $k = 15$, $c = 0.25$, $f = 0.6$
23. $k = 100$, $c = 0.05$, $p = 4$
24. $k = 0.75$, $c = 3$, $p = 3\pi$
25. $k = 7$, $c = 10$, $p = \pi/6$
26. $k = 1$, $c = 1$, $p = 1$
27. $k = 0.3$, $c = 0.2$, $f = 20$
28. $k = 12$, $c = 0.01$, $f = 8$

29–34 ■ Amplitude, Period, Phase, and Horizontal Shift For each sine curve find the amplitude, period, phase, and horizontal shift.

29. $y = 5\sin\left(2t - \frac{\pi}{2}\right)$
30. $y = 10\sin\left(t - \frac{\pi}{3}\right)$
31. $y = 100\sin(5t + \pi)$
32. $y = 50\sin\left(\frac{1}{2}t + \frac{\pi}{5}\right)$
33. $y = 20\sin 2\left(t - \frac{\pi}{4}\right)$
34. $y = 8\sin 4\left(t + \frac{\pi}{12}\right)$

35–38 ■ Phase and Phase Difference A pair of sine curves with the same period is given. **(a)** Find the phase of each curve. **(b)** Find the phase difference between the curves. **(c)** Determine whether the curves are in phase or out of phase. **(d)** Sketch both curves on the same axes.

35. $y_1 = 10\sin\left(3t - \frac{\pi}{2}\right)$; $y_2 = 10\sin\left(3t - \frac{5\pi}{2}\right)$
36. $y_1 = 15\sin\left(2t - \frac{\pi}{3}\right)$; $y_2 = 15\sin\left(2t - \frac{\pi}{6}\right)$

37. $y_1 = 80 \sin 5\left(t - \frac{\pi}{10}\right); \quad y_2 = 80 \sin\left(5t - \frac{\pi}{3}\right)$

38. $y_1 = 20 \sin 2\left(t - \frac{\pi}{2}\right); \quad y_2 = 20 \sin 2\left(t - \frac{3\pi}{2}\right)$

APPLICATIONS

39. A Bobbing Cork A cork floating in a lake is bobbing in simple harmonic motion. Its displacement above the bottom of the lake is modeled by

$$y = 0.2 \cos 20\pi t + 8$$

where y is measured in meters and t is measured in minutes.

(a) Find the frequency of the motion of the cork.

(b) Sketch a graph of y.

(c) Find the maximum displacement of the cork above the lake bottom.

40. FM Radio Signals The carrier wave for an FM radio signal is modeled by the function

$$y = a \sin(2\pi(9.15 \times 10^7)t)$$

where t is measured in seconds. Find the period and frequency of the carrier wave.

41. Blood Pressure Each time your heart beats, your blood pressure increases, then decreases as the heart rests between beats. A certain person's blood pressure is modeled by the function

$$p(t) = 115 + 25 \sin(160\pi t)$$

where $p(t)$ is the pressure (in mmHg) at time t, measured in minutes.

(a) Find the amplitude, period, and frequency of p.

(b) Sketch a graph of p.

(c) If a person is exercising, his or her heart beats faster. How does this affect the period and frequency of p?

42. Predator Population Model In a predator/prey model, the predator population is modeled by the function

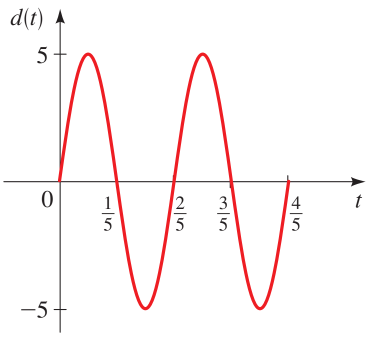
$$y = 900 \cos 2t + 8000$$

where t is measured in years.

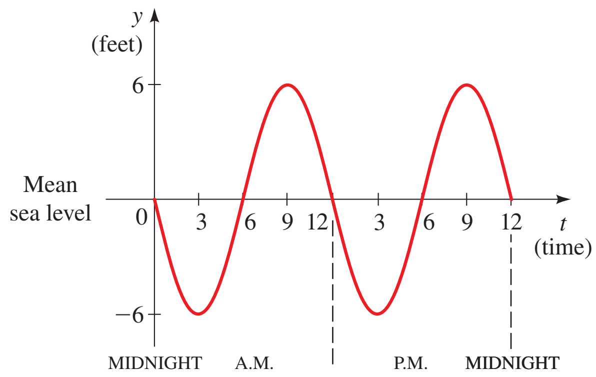
(a) What is the maximum population?

(b) Find the length of time between successive periods of maximum population.

43. Mass-Spring System A mass attached to a spring is moving up and down in simple harmonic motion. The graph gives its displacement $d(t)$ from equilibrium at time t. Express the function d in the form $d(t) = a \sin \omega t$.

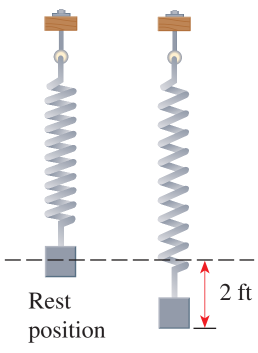


44. Tides The graph shows the variation of the water level relative to mean sea level in Commencement Bay at Tacoma, Washington, for a particular 24-h period. Assuming that this variation is modeled by simple harmonic motion, find an equation of the form $y = a \sin \omega t$ that describes the variation in water level as a function of the number of hours after midnight.



45. Tides The Bay of Fundy in Nova Scotia has the highest tides in the world. In one 12-h period the water starts at mean sea level, rises to 21 ft above, drops to 21 ft below, then returns to mean sea level. Assuming that the motion of the tides is simple harmonic, find an equation that describes the height of the tide in the Bay of Fundy above mean sea level. Sketch a graph that shows the level of the tides over a 12-h period.

46. Mass-Spring System A mass suspended from a spring is pulled down a distance of 2 ft from its rest position, as shown in the figure. The mass is released at time $t = 0$ and allowed to oscillate. If the mass returns to this position after 1 s, find an equation that describes its motion.



47. Mass-Spring System A mass is suspended on a spring. The spring is compressed so that the mass is located 5 cm above its rest position. The mass is released at time $t = 0$ and allowed to oscillate. It is observed that the mass reaches its lowest point $\frac{1}{2}$ s after it is released. Find an equation that describes the motion of the mass.

48. Mass-Spring System The frequency of oscillation of an object suspended on a spring depends on the stiffness k of the spring (called the *spring constant*) and the mass m of the object. If the spring is compressed a distance a and then allowed to oscillate, its displacement is given by

$$f(t) = a\cos\sqrt{k/m}\,t$$

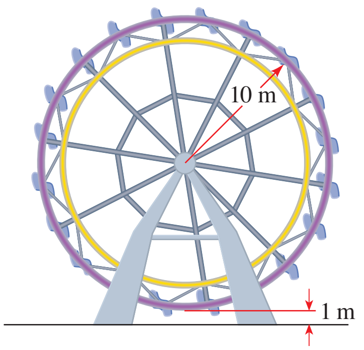
(a) A 10-g mass is suspended from a spring with stiffness $k = 3$. If the spring is compressed a distance 5 cm and then released, find the equation that describes the oscillation of the spring.

(b) Find a general formula for the frequency (in terms of k and m).

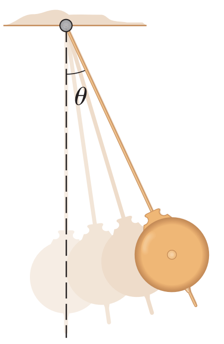
(c) How is the frequency affected if the mass is increased? Is the oscillation faster or slower?

(d) How is the frequency affected if a stiffer spring is used (larger k)? Is the oscillation faster or slower?

49. Ferris Wheel A Ferris wheel has a radius of 10 m, and the bottom of the wheel passes 1 m above the ground. If the Ferris wheel makes one complete revolution every 20 s, find an equation that gives the height above the ground of a person on the Ferris wheel as a function of time.



50. Clock Pendulum The pendulum in a grandfather clock makes one complete swing every 2 s. The maximum angle that the pendulum makes with respect to its rest position is 10°. We know from physical principles that the angle θ between the pendulum and its rest position changes in simple harmonic fashion. Find an equation that describes the size of the angle θ as a function of time. (Take $t = 0$ to be a time when the pendulum is vertical.)



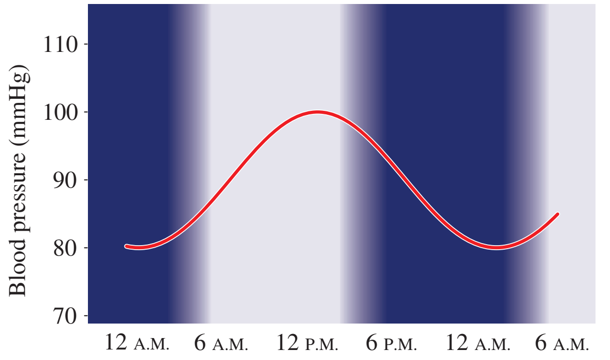
51. Variable Stars The variable star Zeta Gemini has a period of 10 days. The average brightness of the star is 3.8 magnitudes, and the maximum variation from the average is 0.2 magnitude. Assuming that the variation in brightness is simple harmonic, find an equation that gives the brightness of the star as a function of time.

52. Variable Stars Astronomers believe that the radius of a variable star increases and decreases with the brightness of the star. The variable star Delta Cephei (Example 4) has an average radius of 20 million miles and changes by a maximum of 1.5 million miles from this average during a single pulsation. Find an equation that describes the radius of this star as a function of time.

53. Biological Clocks *Circadian rhythms* are biological processes that oscillate with a period of approximately 24 h. That is, a circadian rhythm is an internal daily biological clock. Blood pressure appears to follow such a rhythm. For a certain individual the average resting blood pressure varies from a maximum of 100 mmHg at 2:00 P.M. to a minimum of 80 mmHg at 2:00 A.M. Find a sine function of the form

$$f(t) = a\sin(\omega(t - c)) + b$$

that models the blood pressure at time t, measured in hours from midnight.



54. Electric Generator The armature in an electric generator is rotating at the rate of 100 revolutions per second (rps). If the maximum voltage produced is 310 V, find an equation that describes this variation in voltage. What is the RMS voltage? (See Example 6 and the margin note adjacent to it.)

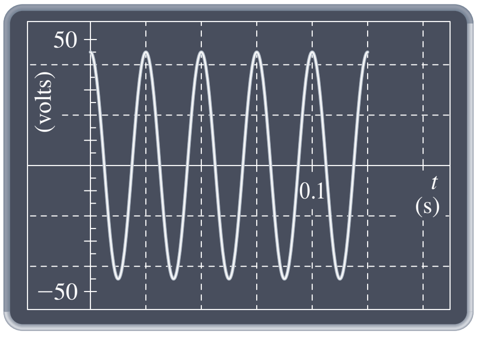
55. Electric Generator The graph shows an oscilloscope reading of the variation in voltage of an AC current produced by a simple generator.

(a) Find the maximum voltage produced.

(b) Find the frequency (cycles per second) of the generator.

(c) How many revolutions per second does the armature in the generator make?

(d) Find a formula that describes the variation in voltage as a function of time.



56. Doppler Effect When a car with its horn blowing drives by an observer, the pitch of the horn seems higher as it approaches and lower as it recedes (see the figure below). This phenomenon is called the **Doppler effect**. If the sound source is moving at speed v relative to the observer and if the speed of sound is v_0, then the perceived frequency f is related to the actual frequency f_0 as follows.

$$f = f_0\left(\frac{v_0}{v_0 \pm v}\right)$$

We choose the minus sign if the source is moving toward the observer and the plus sign if it is moving away.

Suppose that a car drives at 110 ft/s past a woman standing on the shoulder of a highway, blowing its horn, which has a frequency of 500 Hz. Assume that the speed of sound is 1130 ft/s. (This is the speed in dry air at 70°F.)

(a) What are the frequencies of the sounds that the woman hears as the car approaches her and as it moves away from her?

(b) Let A be the amplitude of the sound. Find functions of the form

$$y = A \sin \omega t$$

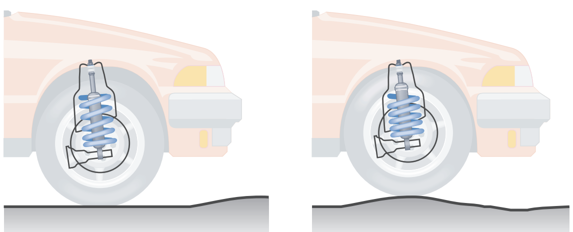
that model the perceived sound as the car approaches the woman and as it recedes.

57. Motion of a Building A strong gust of wind strikes a tall building, causing it to sway back and forth in damped harmonic motion. The frequency of the oscillation is 0.5 cycle per second, and the damping constant is $c = 0.9$. Find an equation that describes the motion of the building. (Assume that $k = 1$, and take $t = 0$ to be the instant when the gust of wind strikes the building.)

58. Shock Absorber When a car hits a certain bump on the road, a shock absorber on the car is compressed a distance of 6 in., then released (see the figure). The shock absorber vibrates in damped harmonic motion with a frequency of 2 cycles per second. The damping constant for this particular shock absorber is 2.8.

(a) Find an equation that describes the displacement of the shock absorber from its rest position as a function of time. Take $t = 0$ to be the instant that the shock absorber is released.

(b) How long does it take for the amplitude of the vibration to decrease to 0.5 in.?



59. Tuning Fork A tuning fork is struck and oscillates in damped harmonic motion. The amplitude of the motion is measured, and 3 s later it is found that the amplitude has dropped to $\frac{1}{4}$ of this value. Find the damping constant c for this tuning fork.

60. Guitar String A guitar string is pulled at point P a distance of 3 cm above its rest position. It is then released and vibrates in damped harmonic motion with a frequency of 165 cycles per second. After 2 s, it is observed that the amplitude of the vibration at point P is 0.6 cm.

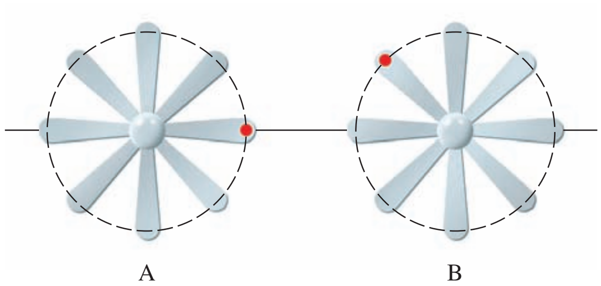
(a) Find the damping constant c.

(b) Find an equation that describes the position of point P above its rest position as a function of time. Take $t = 0$ to be the instant that the string is released.

61. Two Fans Electric fans A and B have radius 1 ft and, when switched on, rotate counterclockwise at the rate of 100 revolutions per minute. Starting with the position shown in the figure, the fans are simultaneously switched on.

(a) For each fan, find an equation that gives the height of the red dot (above the horizontal line shown) t minutes after the fans are switched on.

(b) Are the fans rotating in phase? Through what angle should fan A be rotated counterclockwise in order that the two fans rotate in phase?

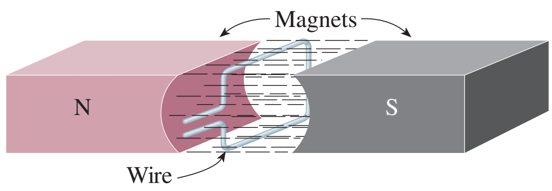


62. Alternating Current Alternating current is produced when an armature rotates about its axis in a magnetic field, as shown in the figure. Generators A and B rotate counterclockwise at 60 Hz (cycles per second) and each generator produces a maximum of 50 V. The voltage for each generator is modeled by

$$E_A = 50\sin(120\pi t) \qquad E_B = 50\sin\left(120\pi t - \frac{5\pi}{4}\right)$$

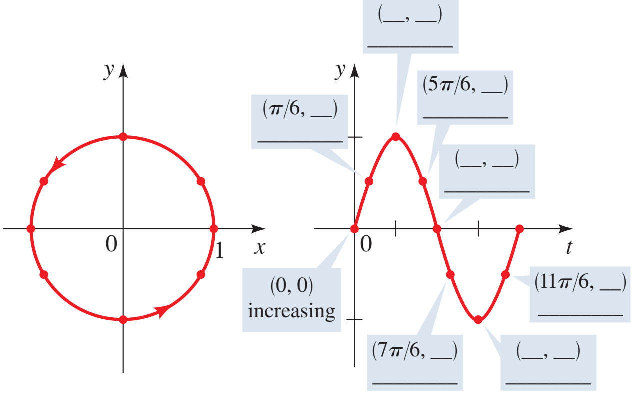
(a) Find the voltage phase for each generator, and find the phase difference.

(b) Are the generators producing voltage in phase? Through what angle should the armature in the second generator be rotated counterclockwise in order that the two generators produce voltage in phase?



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

63. DISCUSS: Phases of Sine The phase of a sine curve $y = \sin(kt + b)$ represents a particular location on the graph of the sine function $y = \sin t$. Specifically, when $t = 0$, we have $y = \sin b$, and this corresponds to the point $(b, \sin b)$ on the graph of $y = \sin t$. Observe that each point on the graph of $y = \sin t$ has different characteristics. For example, for $t = \pi/6$, we have $\sin t = \frac{1}{2}$ and the values of sine are increasing, whereas at $t = 5\pi/6$, we also have $\sin t = \frac{1}{2}$ but the values of sine are decreasing. So each point on the graph of sine corresponds to a different "phase" of a sine curve. Complete the descriptions for each label on the graph below.



64. DISCUSS: Phases of the Moon During the course of a lunar cycle (about 1 month) the moon undergoes the familiar lunar phases. The phases of the moon are completely analogous to the phases of the sine function described in Exercise 63. The figure below shows some phases of the lunar cycle starting with a "new moon," "waxing crescent moon," "first quarter moon," and so on. The next to last phase shown is a "waning crescent moon." Give similar descriptions for the other phases of the moon shown in the figure. What are some events on the earth that follow a monthly cycle and are in phase with the lunar cycle? What are some events that are out of phase with the lunar cycle?

CHAPTER 5 ■ REVIEW

PROPERTIES AND FORMULAS

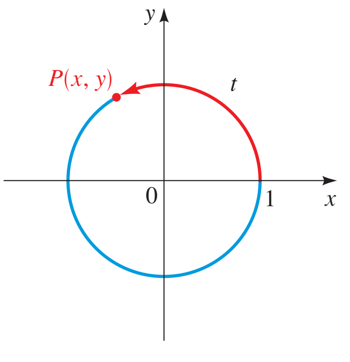
The Unit Circle (p. 402)

The **unit circle** is the circle of radius 1 centered at (0, 0). The equation of the unit circle is $x^2 + y^2 = 1$.

Terminal Points on the Unit Circle (pp. 402–404)

The **terminal point** $P(x, y)$ determined by the real number t is the point obtained by traveling counterclockwise a distance t along the unit circle, starting at $(1, 0)$.

Special terminal points are listed in Table 1 on page 404.



The Reference Number (pp. 405–406)

The **reference number** associated with the real number t is the shortest distance along the unit circle between the terminal point determined by t and the x-axis.

The Trigonometric Functions (p. 409)

Let $P(x, y)$ be the terminal point on the unit circle determined by the real number t. Then for nonzero values of the denominator the trigonometric functions are defined as follows.

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}$$

$$\csc t = \frac{1}{y} \qquad \sec t = \frac{1}{x} \qquad \cot t = \frac{x}{y}$$

Special Values of the Trigonometric Functions (p. 410)

The trigonometric functions have the following values at the special values of t.

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

Basic Trigonometric Identities (pp. 414–415)

An identity is an equation that is true for all values of the variable. The basic trigonometric identities are as follows.

Reciprocal Identities:

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t}$$

Pythagorean Identities:

$$\sin^2 t + \cos^2 t = 1$$
$$\tan^2 t + 1 = \sec^2 t$$
$$1 + \cot^2 t = \csc^2 t$$

Even-Odd Properties:

$$\sin(-t) = -\sin t \qquad \cos(-t) = \cos t \qquad \tan(-t) = -\tan t$$
$$\csc(-t) = -\csc t \qquad \sec(-t) = \sec t \qquad \cot(-t) = -\cot t$$

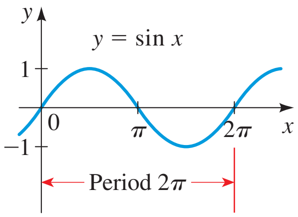
Periodic Properties (p. 419)

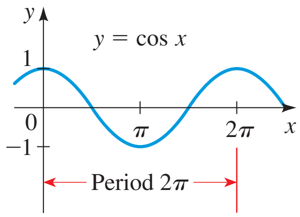
A function f is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every x. The least such p is called the **period** of f. The sine and cosine functions have period 2π, and the tangent function has period π.

$$\sin(t + 2\pi) = \sin t$$
$$\cos(t + 2\pi) = \cos t$$
$$\tan(t + \pi) = \tan t$$

Graphs of the Sine and Cosine Functions (p. 420)

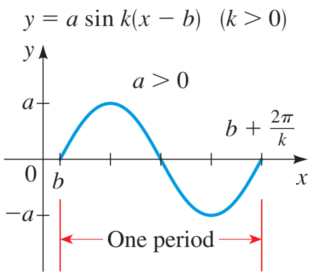
The graphs of sine and cosine have amplitude 1 and period 2π.

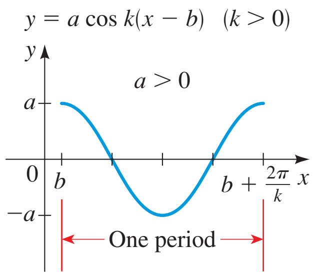




Amplitude 1, Period 2π

Graphs of Transformations of Sine and Cosine (p. 424)



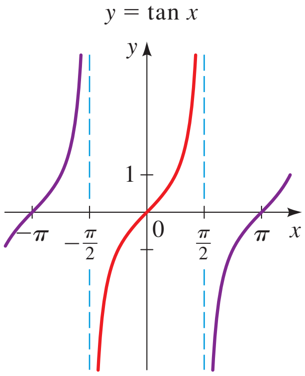


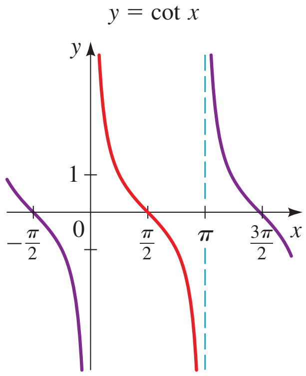
Amplitude a, Period $\frac{2\pi}{k}$, Horizontal shift b

An appropriate interval on which to graph one complete period is $[b, b + (2\pi/k)]$.

Graphs of the Tangent and Cotangent Functions (pp. 434–435)

These functions have period π.



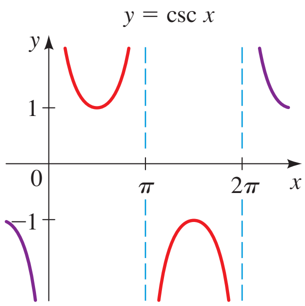


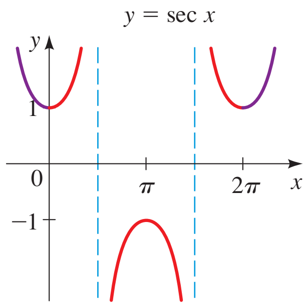
To graph one period of $y = a \tan kx$, an appropriate interval is $(-\pi/2k, \pi/2k)$.

To graph one period of $y = a \cot kx$, an appropriate interval is $(0, \pi/k)$.

Graphs of the Cosecant and Secant Functions (pp. 436–437)

These functions have period 2π.





To graph one period of $y = a \csc kx$, an appropriate interval is $(0, 2\pi/k)$.

To graph one period of $y = a \sec kx$, an appropriate interval is $(0, 2\pi/k)$.

Inverse Trigonometric Functions (pp. 440–443)

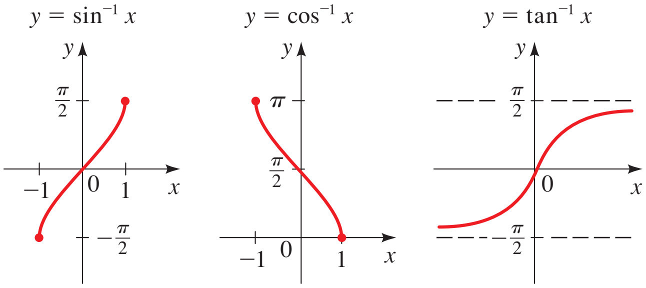
Inverse functions of the trigonometric functions are defined by restricting the domains as follows.

Function	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

The inverse trigonometric functions are defined as follows.

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$
$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$
$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

Graphs of these inverse functions are shown below.



Harmonic Motion (p. 446)

An object is in **simple harmonic motion** if its displacement y at time t is modeled by $y = a \sin \omega t$ or $y = a \cos \omega t$. In this case the amplitude is $|a|$, the period is $2\pi/\omega$, and the frequency is $\omega/2\pi$.

Damped Harmonic Motion (p. 451)

An object is in **damped harmonic motion** if its displacement y at time t is modeled by $y = ke^{-ct} \sin \omega t$ or $y = ke^{-ct} \cos \omega t$, $c > 0$. In this case c is the damping constant, k is the initial amplitude, and $2\pi/\omega$ is the period.

Phase (pp. 453–454)

Any sine curve can be expressed in the following equivalent forms:

$y = A \sin(kt - b)$, the **phase** is b

$y = A \sin k\left(t - \dfrac{b}{k}\right)$, the **horizontal shift** is $\dfrac{b}{k}$

The phase (or phase angle) b is the initial angular position of the motion. The number b/k is also called the **lag time** ($b > 0$) or **lead time** ($b < 0$).

Suppose that two objects are in harmonic motion with the same period modeled by

$$y_1 = A \sin(kt - b) \quad \text{and} \quad y_2 = A \sin(kt - c)$$

The **phase difference** between y_1 and y_2 is $b - c$. The motions are "in phase" if the phase difference is a multiple of 2π; otherwise, the motions are "out of phase."

CONCEPT CHECK

1. **(a)** What is the unit circle, and what is the equation of the unit circle?
   (b) Use a diagram to explain what is meant by the terminal point $P(x, y)$ determined by t.
   (c) Find the terminal point for $t = \dfrac{\pi}{2}$.
   (d) What is the reference number associated with t?
   (e) Find the reference number and terminal point for $t = \dfrac{7\pi}{4}$.

2. Let t be a real number, and let $P(x, y)$ be the terminal point determined by t.
   (a) Write equations that define $\sin t$, $\cos t$, $\tan t$, $\csc t$, $\sec t$, and $\cot t$.
   (b) In each of the four quadrants, identify the trigonometric functions that are positive.
   (c) List the special values of sine, cosine, and tangent.

3. **(a)** Describe the steps we use to find the value of a trigonometric function at a real number t.
   (b) Find $\sin \dfrac{5\pi}{6}$.

4. **(a)** What is a periodic function?
   (b) What are the periods of the six trigonometric functions?
   (c) Find $\sin \dfrac{19\pi}{4}$.

5. **(a)** What is an even function, and what is an odd function?
   (b) Which trigonometric functions are even? Which are odd?
   (c) If $\sin t = 0.4$, find $\sin(-t)$.
   (d) If $\cos s = 0.7$, find $\cos(-s)$.

6. **(a)** State the reciprocal identities.
   (b) State the Pythagorean identities.

7. **(a)** Graph the sine and cosine functions.
   (b) What are the amplitude, period, and horizontal shift for the sine curve $y = a \sin k(x - b)$ and for the cosine curve $y = a \cos k(x - b)$?
   (c) Find the amplitude, period, and horizontal shift of $y = 3 \sin\left(2x - \dfrac{\pi}{6}\right)$.

8. **(a)** Graph the tangent and cotangent functions.
   (b) For the curves $y = a \tan kx$ and $y = a \cot kx$, state appropriate intervals to graph one complete period of each curve.
   (c) Find an appropriate interval to graph one complete period of $y = 5 \tan 3x$.

9. **(a)** Graph the cosecant and secant functions.
   (b) For the curves $y = a \csc kx$ and $y = a \sec kx$, state appropriate intervals to graph one complete period of each curve.
   (c) Find an appropriate interval to graph one period of $y = 3 \csc 6x$.

10. **(a)** Define the inverse sine function, the inverse cosine function, and the inverse tangent function.
    (b) Find $\sin^{-1} \dfrac{1}{2}$, $\cos^{-1} \dfrac{\sqrt{2}}{2}$, and $\tan^{-1} 1$.
    (c) For what values of x is the equation $\sin(\sin^{-1} x) = x$ true? For what values of x is the equation $\sin^{-1}(\sin x) = x$ true?

11. **(a)** What is simple harmonic motion?
    (b) What is damped harmonic motion?
    (c) Give real-world examples of harmonic motion.

12. Suppose that an object is in simple harmonic motion given by $y = 5\sin\left(2t - \frac{\pi}{3}\right)$.

(a) Find the amplitude, period, and frequency.

(b) Find the phase and the horizontal shift.

13. Consider the following models of harmonic motion.

$$y_1 = 5\sin(2t - 1) \qquad y_2 = 5\sin(2t - 3)$$

Do both motions have the same frequency? What is the phase for each equation? What is the phase difference? Are the objects moving in phase or out of phase?

ANSWERS TO THE CONCEPT CHECK CAN BE FOUND AT THE BACK OF THE BOOK.

EXERCISES

1–2 ■ Terminal Points A point $P(x, y)$ is given. **(a)** Show that P is on the unit circle. **(b)** Suppose that P is the terminal point determined by t. Find $\sin t$, $\cos t$, and $\tan t$.

1. $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

2. $P\left(\frac{3}{5}, -\frac{4}{5}\right)$

3–6 ■ Reference Number and Terminal Point A real number t is given. **(a)** Find the reference number for t. **(b)** Find the terminal point $P(x, y)$ on the unit circle determined by t. **(c)** Find the six trigonometric functions of t.

3. $t = \frac{2\pi}{3}$

4. $t = \frac{5\pi}{3}$

5. $t = -\frac{11\pi}{4}$

6. $t = -\frac{7\pi}{6}$

7–16 ■ Values of Trigonometric Functions Find the value of the trigonometric function. If possible, give the exact value; otherwise, use a calculator to find an approximate value rounded to five decimal places.

7. (a) $\sin\frac{3\pi}{4}$ **(b)** $\cos\frac{3\pi}{4}$

8. (a) $\tan\frac{\pi}{3}$ **(b)** $\tan\left(-\frac{\pi}{3}\right)$

9. (a) $\sin 1.1$ **(b)** $\cos 1.1$

10. (a) $\cos\frac{\pi}{5}$ **(b)** $\cos\left(-\frac{\pi}{5}\right)$

11. (a) $\cos\frac{9\pi}{2}$ **(b)** $\sec\frac{9\pi}{2}$

12. (a) $\sin\frac{\pi}{7}$ **(b)** $\csc\frac{\pi}{7}$

13. (a) $\tan\frac{5\pi}{2}$ **(b)** $\cot\frac{5\pi}{2}$

14. (a) $\sin 2\pi$ **(b)** $\csc 2\pi$

15. (a) $\tan\frac{5\pi}{6}$ **(b)** $\cot\frac{5\pi}{6}$

16. (a) $\cos\frac{\pi}{3}$ **(b)** $\sin\frac{\pi}{6}$

17–20 ■ Fundamental Identities Use the fundamental identities to write the first expression in terms of the second.

17. $\frac{\tan t}{\cos t}$, $\sin t$

18. $\tan^2 t \sec t$, $\cos t$

19. $\tan t$, $\sin t$; t in Quadrant IV

20. $\sec t$, $\sin t$; t in Quadrant II

21–24 ■ Values of Trigonometric Functions Find the values of the remaining trigonometric functions at t from the given information.

21. $\sin t = \frac{5}{13}$, $\cos t = -\frac{12}{13}$

22. $\sin t = -\frac{1}{2}$, $\cos t > 0$

23. $\cot t = -\frac{1}{2}$, $\csc t = \sqrt{5}/2$

24. $\cos t = -\frac{3}{5}$, $\tan t < 0$

25–28 ■ Values of Trigonometric Functions Find the values of the trigonometric function of t from the given information.

25. $\sec t + \cot t$; $\tan t = \frac{1}{4}$, terminal point for t in Quadrant III

26. $\csc t + \sec t$; $\sin t = -\frac{8}{17}$, terminal point for t in Quadrant IV

27. $\tan t + \sec t$; $\cos t = \frac{3}{5}$, terminal point for t in Quadrant I

28. $\sin^2 t + \cos^2 t$; $\sec t = -5$, terminal point for t in Quadrant II

29–36 ■ Horizontal Shifts A trigonometric function is given. **(a)** Find the amplitude, period, and horizontal shift of the function. **(b)** Sketch the graph.

29. $y = 10\cos\frac{1}{2}x$

30. $y = 4\sin 2\pi x$

31. $y = -\sin\frac{1}{2}x$

32. $y = 2\sin\left(x - \frac{\pi}{4}\right)$

33. $y = 3\sin(2x - 2)$

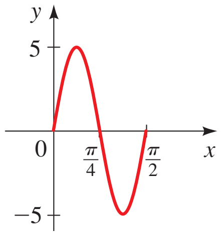
34. $y = \cos 2\left(x - \frac{\pi}{2}\right)$

35. $y = -\cos\left(\frac{\pi}{2}x + \frac{\pi}{6}\right)$

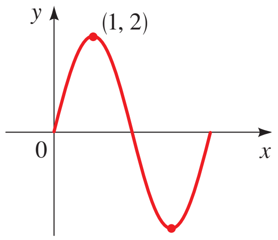
36. $y = 10\sin\left(2x - \frac{\pi}{2}\right)$

37–40 ■ Functions from a Graph The graph of one period of a function of the form $y = a \sin k(x - b)$ or $y = a \cos k(x - b)$ is shown. Determine the function.

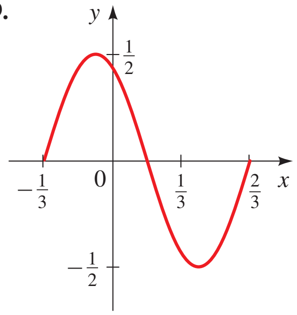
37.



38.



39.



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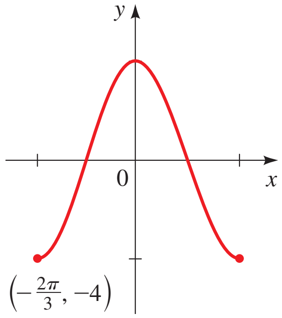


41–48 ■ Graphing Trigonometric Functions Find the period, and sketch the graph.

41. $y = 3 \tan x$

42. $y = \tan \pi x$

43. $y = 2 \cot\left(x - \dfrac{\pi}{2}\right)$

44. $y = \sec\left(\dfrac{1}{2}x - \dfrac{\pi}{2}\right)$

45. $y = 4 \csc(2x + \pi)$

46. $y = \tan\left(x + \dfrac{\pi}{6}\right)$

47. $y = \tan\left(\dfrac{1}{2}x - \dfrac{\pi}{8}\right)$

48. $y = -4 \sec 4\pi x$

49–52 ■ Evaluating Expressions Involving Inverse Trigonometric Functions Find the exact value of each expression, if it is defined.

49. $\sin^{-1} 1$

50. $\cos^{-1}\left(-\frac{1}{2}\right)$

51. $\sin^{-1}\left(\sin \dfrac{13\pi}{6}\right)$

52. $\tan\left(\cos^{-1}\left(\dfrac{1}{2}\right)\right)$

53–54 ■ Amplitude, Period, Phase, and Horizontal Shift For each sine curve find the amplitude, period, phase, and horizontal shift.

53. $y = 100 \sin 8\left(t + \dfrac{\pi}{16}\right)$

54. $y = 80 \sin 3\left(t - \dfrac{\pi}{2}\right)$

55–56 ■ Phase and Phase Difference A pair of sine curves with the same period is given. **(a)** Find the phase of each curve. **(b)** Find the phase difference between the curves. **(c)** Determine whether the curves are in phase or out of phase. **(d)** Sketch both curves on the same axes.

55. $y_1 = 25 \sin 3\left(t - \dfrac{\pi}{2}\right)$; $y_2 = 10 \sin\left(3t - \dfrac{5\pi}{2}\right)$

56. $y_1 = 50 \sin\left(10t - \dfrac{\pi}{2}\right)$; $y_2 = 50 \sin 10\left(t - \dfrac{\pi}{20}\right)$

57–62 ■ Even and Odd Functions A function is given. **(a)** Use a graphing device to graph the function. **(b)** Determine from the graph whether the function is periodic and, if so, determine the period. **(c)** Determine from the graph whether the function is odd, even, or neither.

57. $y = |\cos x|$

58. $y = \sin(\cos x)$

59. $y = \cos(2^{0.1x})$

60. $y = 1 + 2^{\cos x}$

61. $y = |x| \cos 3x$

62. $y = \sqrt{x} \sin 3x, \quad x > 0$

63–66 ■ Sine and Cosine Curves with Variable Amplitude Graph the three functions on a common screen. How are the graphs related?

63. $y = x, \quad y = -x, \quad y = x \sin x$

64. $y = 2^{-x}, \quad y = -2^{-x}, \quad y = 2^{-x} \cos 4\pi x$

65. $y = x, \quad y = \sin 4x, \quad y = x + \sin 4x$

66. $y = \sin^2 x, \quad y = \cos^2 x, \quad y = \sin^2 x + \cos^2 x$

67–68 ■ Maxima and Minima Find the maximum and minimum values of the function.

67. $y = \cos x + \sin 2x$

68. $y = \cos x + \sin^2 x$

69-70 ■ Solving Trigonometric Equations Graphically Find all solutions of the equation that lie in the given interval. State each answer rounded to two decimal places.

69. $\sin x = 0.3; \quad [0, 2\pi]$

70. $\cos 3x = x; \quad [0, \pi]$

71. Discover the Period of a Trigonometric Function Let $y_1 = \cos(\sin x)$ and $y_2 = \sin(\cos x)$.

(a) Graph y_1 and y_2 in the same viewing rectangle.

(b) Determine the period of each of these functions from its graph.

(c) Find an inequality between $\sin(\cos x)$ and $\cos(\sin x)$ that is valid for all x.

72. Simple Harmonic Motion A point P moving in simple harmonic motion completes 8 cycles every second. If the amplitude of the motion is 50 cm, find an equation that describes the motion of P as a function of time. Assume that the point P is at its maximum displacement when $t = 0$.

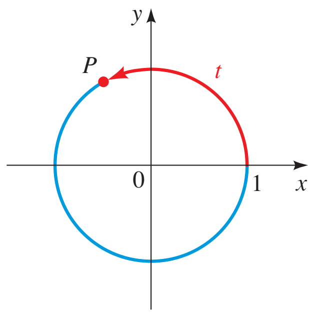
73. Simple Harmonic Motion A mass suspended from a spring oscillates in simple harmonic motion at a frequency of 4 cycles per second. The distance from the highest to the lowest point of the oscillation is 100 cm. Find an equation that describes the distance of the mass from its rest position as a function of time. Assume that the mass is at its lowest point when $t = 0$.

74. Damped Harmonic Motion The top floor of a building undergoes damped harmonic motion after a sudden brief earthquake. At time $t = 0$ the displacement is at a maximum, 16 cm from the normal position. The damping constant is $c = 0.72$, and the building vibrates at 1.4 cycles per second.

(a) Find a function of the form $y = ke^{-ct} \cos \omega t$ to model the motion.

(b) Graph the function you found in part (a).

(c) What is the displacement at time $t = 10$ s?



1. The point $P(x, y)$ is on the unit circle in Quadrant IV. If $x = \sqrt{11}/6$, find y.

2. The point P in the figure at the left has y-coordinate $\frac{4}{5}$. Find:
 (a) $\sin t$ **(b)** $\cos t$
 (c) $\tan t$ **(d)** $\sec t$

3. Find the exact value.
 (a) $\sin \frac{7\pi}{6}$ **(b)** $\cos \frac{13\pi}{4}$
 (c) $\tan\left(-\frac{5\pi}{3}\right)$ **(d)** $\csc \frac{3\pi}{2}$

4. Express $\tan t$ in terms of $\sin t$, if the terminal point determined by t is in Quadrant II.

5. If $\cos t = -\frac{8}{17}$ and if the terminal point determined by t is in Quadrant III, find $\tan t \cot t + \csc t$.

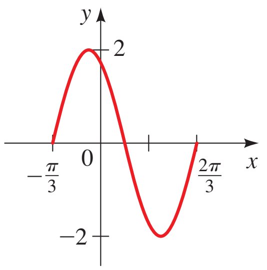
6–7 ■ A trigonometric function is given.
 (a) Find the amplitude, period, phase, and horizontal shift of the function.
 (b) Sketch the graph of one complete period.

6. $y = -5 \cos 4x$ 7. $y = 2 \sin\left(\frac{1}{2}x - \frac{\pi}{6}\right)$

8–9 ■ Find the period, and graph the function.

8. $y = -\csc 2x$ 9. $y = \tan\left(2x - \frac{\pi}{2}\right)$

10. Find the exact value of each expression, if it is defined.
 (a) $\tan^{-1} 1$ **(b)** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
 (c) $\tan^{-1}(\tan 3\pi)$ **(d)** $\cos(\tan^{-1}(-\sqrt{3}))$



11. The graph shown at left is one period of a function of the form $y = a \sin k(x - b)$. Determine the function.

12. The sine curves $y_1 = 30 \sin\left(6t - \frac{\pi}{2}\right)$ and $y_2 = 30 \sin\left(6t - \frac{\pi}{3}\right)$ have the same period.
 (a) Find the phase of each curve.
 (b) Find the phase difference between y_1 and y_2.
 (c) Determine whether the curves are in phase or out of phase.
 (d) Sketch both curves on the same axes.

13. Let $f(x) = \frac{\cos x}{1 + x^2}$.
 (a) Use a graphing device to graph f in an appropriate viewing rectangle.
 (b) Determine from the graph if f is even, odd, or neither.
 (c) Find the minimum and maximum values of f.

14. A mass suspended from a spring oscillates in simple harmonic motion. The mass completes 2 cycles every second, and the distance between the highest point and the lowest point of the oscillation is 10 cm. Find an equation of the form $y = a \sin \omega t$ that gives the distance of the mass from its rest position as a function of time.

15. An object is moving up and down in damped harmonic motion. Its displacement at time $t = 0$ is 16 in.; this is its maximum displacement. The damping constant is $c = 0.1$, and the frequency is 12 Hz.
 (a) Find a function that models this motion.
 (b) Graph the function.

FOCUS ON MODELING Fitting Sinusoidal Curves to Data

In previous *Focus on Modeling* sections, we learned how to fit linear, polynomial, exponential, and power models to data. Figure 1 shows some scatter plots of data. The scatter plots can help guide us in choosing an appropriate model. (Try to determine what type of function would best model the data in each graph.) If the scatter plot indicates simple harmonic motion, then we might try to model the data with a sine or cosine function. The next example illustrates this process.

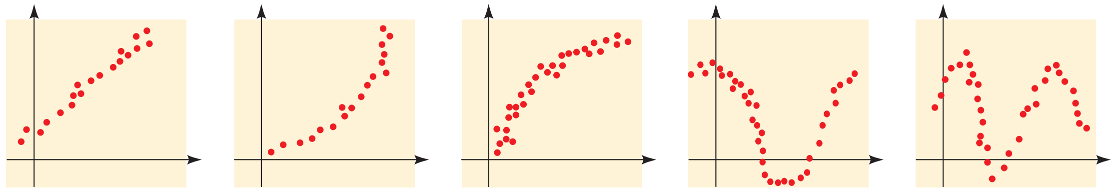


FIGURE 1

EXAMPLE 1 ■ Modeling the Height of a Tide

The water depth in a narrow channel varies with the tides. Table 1 shows the water depth over a 12-h period. A scatter plot of the data is shown in Figure 2.

(a) Find a function that models the water depth with respect to time.

(b) If a boat needs at least 11 ft of water to cross the channel, during which times can it safely do so?

TABLE 1

Time	Depth (ft)
12:00 A.M.	9.8
1:00 A.M.	11.4
2:00 A.M.	11.6
3:00 A.M.	11.2
4:00 A.M.	9.6
5:00 A.M.	8.5
6:00 A.M.	6.5
7:00 A.M.	5.7
8:00 A.M.	5.4
9:00 A.M.	6.0
10:00 A.M.	7.0
11:00 A.M.	8.6
12:00 P.M.	10.0

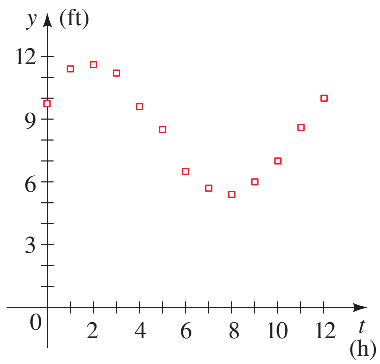


FIGURE 2

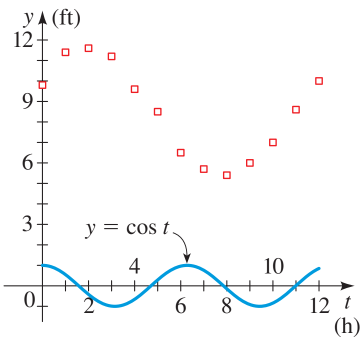


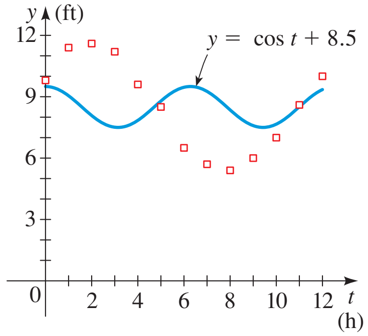
FIGURE 3

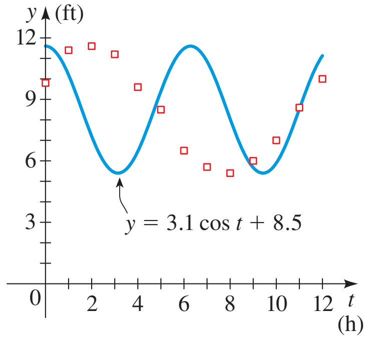
SOLUTION

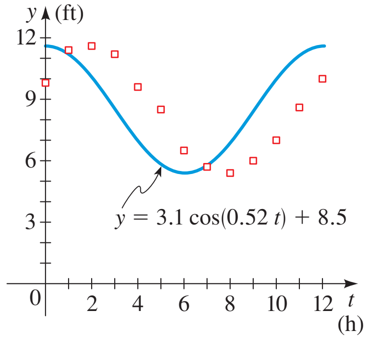
(a) The data appear to lie on a cosine (or sine) curve. But if we graph $y = \cos t$ on the same graph as the scatter plot, the result in Figure 3 is not even close to the data. To fit the data, we need to adjust the vertical shift, amplitude, period, and phase shift of the cosine curve. In other words, we need to find a function of the form

$$y = a\cos(\omega(t - c)) + b$$

We use the following steps, which are illustrated by the graphs in the margin on the next page.







■ **Adjust the Vertical Shift** The vertical shift b is the average of the maximum and minimum values:

$$\begin{aligned} b &= \text{vertical shift} \\ &= \frac{1}{2}\cdot(\text{maximum value} + \text{minimum value}) \\ &= \frac{1}{2}(11.6 + 5.4) = 8.5 \end{aligned}$$

■ **Adjust the Amplitude** The amplitude a is half of the difference between the maximum and minimum values:

$$\begin{aligned} a &= \text{amplitude} \\ &= \frac{1}{2}\cdot(\text{maximum value} - \text{minimum value}) \\ &= \frac{1}{2}(11.6 - 5.4) = 3.1 \end{aligned}$$

■ **Adjust the Period** The time between consecutive maximum and minimum values is half of one period. Thus

$$\begin{aligned} \frac{2\pi}{\omega} &= \text{period} \\ &= 2\cdot(\text{time of maximum value} - \text{time of minimum value}) \\ &= 2(8 - 2) = 12 \end{aligned}$$

Thus $\omega = 2\pi/12 = 0.52$.

■ **Adjust the Horizontal Shift** Since the maximum value of the data occurs at approximately $t = 2.0$, it represents a cosine curve shifted 2 h to the right. So

$$\begin{aligned} c &= \text{phase shift} \\ &= \text{time of maximum value} \\ &= 2.0 \end{aligned}$$

■ **The Model** We have shown that a function that models the tides over the given time period is given by

$$y = 3.1\cos(0.52(t - 2.0)) + 8.5$$

A graph of the function and the scatter plot are shown in Figure 4. It appears that the model we found is a good approximation to the data.

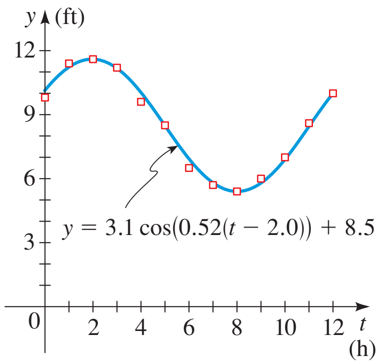


FIGURE 4

(b) We need to solve the inequality $y \geq 11$. We solve this inequality graphically by graphing $y = 3.1 \cos 0.52(t - 2.0) + 8.5$ and $y = 11$ on the same graph. From the graph in Figure 5 we see the water depth is higher than 11 ft between $t \approx 0.8$ and $t \approx 3.2$. This corresponds to the times 12:48 A.M. to 3:12 A.M.

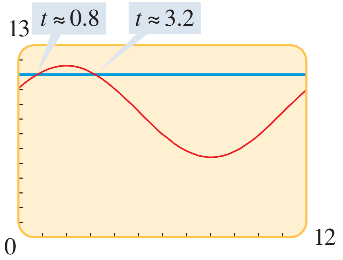


FIGURE 5

For the TI-83 and TI-84 the command `SinReg` (for sine regression) finds the sine curve that best fits the given data.

In Example 1 we used the scatter plot to guide us in finding a cosine curve that gives an approximate model of the data. Some graphing calculators are capable of finding a sine or cosine curve that best fits a given set of data points. The method these calculators use is similar to the method of finding a line of best fit, as explained on page 140.

EXAMPLE 2 ■ Fitting a Sine Curve to Data

(a) Use a graphing device to find the sine curve that best fits the depth of water data in Table 1 on page 466.

(b) Compare your result to the model found in Example 1.

SOLUTION

```
SinReg
 y=a*sin(bx+c)+d
 a=3.097877596
 b=.5268322697
 c=.5493035195
 d=8.424021899
```

Output of the `SinReg` function on the TI-83.

(a) Using the data in Table 1 and the `SinReg` command on the TI-83 calculator, we get a function of the form

$$y = a \sin(bt + c) + d$$

where

$$a = 3.1 \qquad b = 0.53$$

$$c = 0.55 \qquad d = 8.42$$

So the sine function that best fits the data is

$$y = 3.1 \sin(0.53t + 0.55) + 8.42$$

(b) To compare this with the function in Example 1, we change the sine function to a cosine function by using the reduction formula $\sin u = \cos(u - \pi/2)$.

$$y = 3.1 \sin(0.53t + 0.55) + 8.42$$

$$= 3.1 \cos\left(0.53t + 0.55 - \frac{\pi}{2}\right) + 8.42 \qquad \text{Reduction formula}$$

$$= 3.1 \cos(0.53t - 1.02) + 8.42$$

$$= 3.1 \cos(0.53(t - 1.92)) + 8.42 \qquad \text{Factor 0.53}$$

Comparing this with the function we obtained in Example 1, we see that there are small differences in the coefficients. In Figure 6 we graph a scatter plot of the data together with the sine function of best fit.

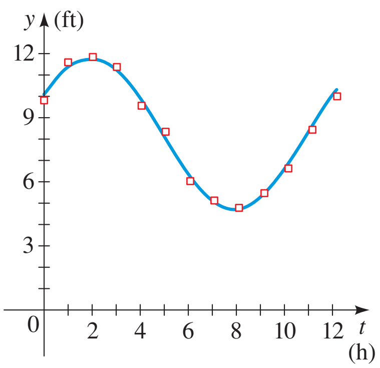


FIGURE 6

In Example 1 we estimated the values of the amplitude, period, and shifts from the data. In Example 2 the calculator computed the sine curve that best fits the data (that is, the curve that deviates least from the data as explained on page 140). The different ways of obtaining the model account for the differences in the functions.

PROBLEMS

1–4 ■ Modeling Periodic Data A set of data is given.

(a) Make a scatter plot of the data.

(b) Find a cosine function of the form $y = a\cos(\omega(t - c)) + b$ that models the data, as in Example 1.

(c) Graph the function you found in part (b) together with the scatter plot. How well does the curve fit the data?

(d) Use a graphing calculator to find the sine function that best fits the data, as in Example 2.

(e) Compare the functions you found in parts (b) and (d). [Use the reduction formula $\sin u = \cos(u - \pi/2)$.]

1.

t	y
0	2.1
2	1.1
4	−0.8
6	−2.1
8	−1.3
10	0.6
12	1.9
14	1.5

2.

t	y
0	190
25	175
50	155
75	125
100	110
125	95
150	105
175	120
200	140
225	165
250	185
275	200
300	195
325	185
350	165

3.

t	y
0.1	21.1
0.2	23.6
0.3	24.5
0.4	21.7
0.5	17.5
0.6	12.0
0.7	5.6
0.8	2.2
0.9	1.0
1.0	3.5
1.1	7.6
1.2	13.2
1.3	18.4
1.4	23.0
1.5	25.1

4.

t	y
0.0	0.56
0.5	0.45
1.0	0.29
1.5	0.13
2.0	0.05
2.5	−0.10
3.0	0.02
3.5	0.12
4.0	0.26
4.5	0.43
5.0	0.54
5.5	0.63
6.0	0.59

5. Circadian Rhythms Circadian rhythm (from the Latin *circa*—about, and *diem*—day) is the daily biological pattern by which body temperature, blood pressure, and other physiological variables change. The data in the table below show typical changes in human body temperature over a 24-h period ($t = 0$ corresponds to midnight).

(a) Make a scatter plot of the data.

(b) Find a cosine curve that models the data (as in Example 1).

(c) Graph the function you found in part (b) together with the scatter plot.

(d) Use a graphing calculator to find the sine curve that best fits the data (as in Example 2).

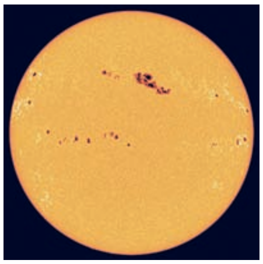
Time	Body temperature (°C)	Time	Body temperature (°C)
0	36.8	14	37.3
2	36.7	16	37.4
4	36.6	18	37.3
6	36.7	20	37.2
8	36.8	22	37.0
10	37.0	24	36.8
12	37.2		

Year	Owl population
0	50
1	62
2	73
3	80
4	71
5	60
6	51
7	43
8	29
9	20
10	28
11	41
12	49

6. **Predator Population** When two species interact in a predator/prey relationship, the populations of both species tend to vary in a sinusoidal fashion. (See *Discovery Project: Predator/Prey Models* referenced on page 427). In a certain midwestern county, the main food source for barn owls consists of field mice and other small mammals. The table gives the population of barn owls in this county every July 1 over a 12-year period.
   (a) Make a scatter plot of the data.
   (b) Find a sine curve that models the data (as in Example 1).
   (c) Graph the function you found in part (b) together with the scatter plot.
   (d) Use a graphing calculator to find the sine curve that best fits the data (as in Example 2). Compare to your answer from part (b).

7. **Salmon Survival** For reasons that are not yet fully understood, the number of fingerling salmon that survive the trip from their riverbed spawning grounds to the open ocean varies approximately sinusoidally from year to year. The table shows the number of salmon that hatch in a certain British Columbia creek and then make their way to the Strait of Georgia. The data are given in thousands of fingerlings, over a period of 16 years.
   (a) Make a scatter plot of the data.
   (b) Find a sine curve that models the data (as in Example 1).
   (c) Graph the function you found in part (b) together with the scatter plot.
   (d) Use a graphing calculator to find the sine curve that best fits the data (as in Example 2). Compare to your answer from part (b).

Year	Salmon (× 1000)	Year	Salmon (× 1000)
1985	43	1993	56
1986	36	1994	63
1987	27	1995	57
1988	23	1996	50
1989	26	1997	44
1990	33	1998	38
1991	43	1999	30
1992	50	2000	22



Science Source

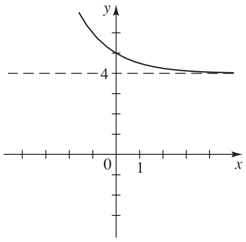
8. **Sunspot Activity** Sunspots are relatively "cool" regions on the sun that appear as dark spots when observed through special solar filters. The number of sunspots varies in an 11-year cycle. The table gives the average daily sunspot count for the years 1968–2012.
   (a) Make a scatter plot of the data.
   (b) Find a cosine curve that models the data (as in Example 1).
   (c) Graph the function you found in part (b) together with the scatter plot.
   (d) Use a graphing calculator to find the sine curve that best fits the data (as in Example 2). Compare to your answer in part (b).

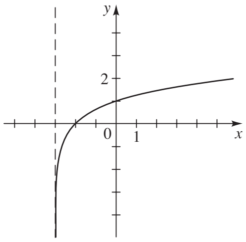
Year	Sunspots	Year	Sunspots	Year	Sunspots	Year	Sunspots
1968	106	1980	154	1991	145	2002	104
1969	105	1981	140	1992	94	2003	63
1970	104	1982	115	1993	54	2004	40
1971	67	1983	66	1994	29	2005	30
1972	69	1984	45	1995	17	2006	15
1973	38	1985	17	1996	8	2007	7
1974	34	1986	13	1997	21	2008	3
1975	15	1987	29	1998	64	2009	3
1976	12	1988	100	1999	93	2010	16
1977	27	1989	157	2000	119	2011	56
1978	92	1990	142	2001	111	2012	58
1979	155						

Source: Solar Influence Data Analysis Center, Belgium

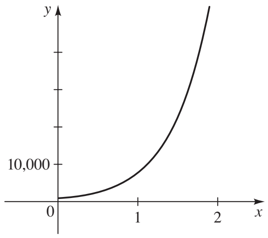
CHAPTER 4 TEST ■ PAGE 391

1. (a) $\mathbb{R}$, $(4, \infty)$, $y = 4$ **(b)** $(-3, \infty)$, $\mathbb{R}$, $x = -3$





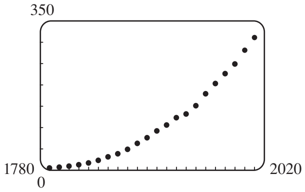
2. (a) $\left(\frac{3}{2}, \infty\right)$ **(b)** $(-\infty, -1) \cup (1, \infty)$
3. (a) $\log_6 25 = 2x$ **(b)** $e^3 = A$
4. (a) 36 **(b)** 3 **(c)** $\frac{3}{2}$ **(d)** 3 **(e)** $\frac{2}{3}$ **(f)** 2
5. (a) $\log x + 3 \log y - 2 \log z$ **(b)** $\frac{1}{2} \ln x - \frac{1}{2} \ln y$
(c) $\frac{1}{3}[\log(x + 2) - 4 \log x - \log(x^2 + 4)]$
6. (a) $\log(ab^2)$ **(b)** $\ln(x - 5)$ **(c)** $\log_2 \dfrac{3\sqrt{x + 1}}{x^3}$
7. (a) 25 **(b)** 1, 2 **(c)** 11.13 **(d)** 5.39
8. (a) 500 **(b)** $\frac{2}{3}$ **(c)** $3 - e^{4/5} \approx 0.774$ **(d)** 2 **9.** 1.326
10. (a) $n(t) = 1000e^{2.07944t}$ **(b)** 22,600 **(c)** 1.3
(d)



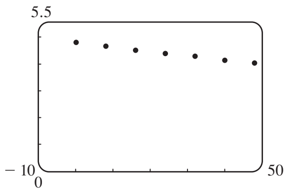
11. (a) $A(t) = 12{,}000\left(1 + \dfrac{0.056}{12}\right)^{12t}$ **(b)** \$14,195.06
(c) 9.12 years **12. (a)** $m(t) = 3 \cdot 2^{-t/10}$ **(b)** $m(t) = 3e^{-0.0693t}$
(c) 0.047 g **(d)** after 3.6 min **13.** 1995 times more intense

FOCUS ON MODELING ■ PAGE 398

1. (a)

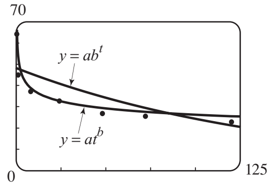


(b) $y = ab^t$, where $a = 3.334926 \times 10^{-15}$, $b = 1.019844$, and y is the population in millions in the year t **(c)** 577 million
(d) 196 million
3. (a)

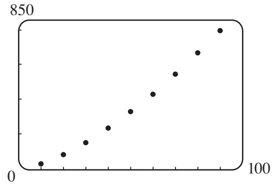


(b) $y = ab^t$, where $a = 4.79246$ and $b = 0.99642$ **(c)** 192.8 h

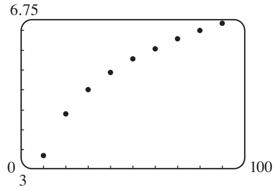
5. (a) $y = at^b$, where $a = 49.70030$ and $b = -0.15437$; $y = ab^t$, where $a = 44.82418$ and $b = 0.99317$
(b)

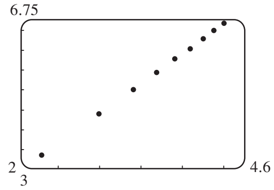


(c) The power function
7. $y = ab^x$, where $a = 2.414$ and $b = 1.05452$
9. (a)



(b)





(c) The power function
(d) $y = ax^b$, where $a = 0.893421326$ and $b = 1.50983$

CHAPTER 5

SECTION 5.1 ■ PAGE 407

1. (a) $(0, 0)$, 1 **(b)** $x^2 + y^2 = 1$ **(c)** (i) 0 (ii) 0 (iii) 0 (iv) 0 **2. (a)** terminal **(b)** $(0, 1), (-1, 0), (0, -1), (1, 0)$
9. $-\frac{4}{5}$ **11.** $-2\sqrt{2}/3$ **13.** $3\sqrt{5}/7$ **15.** $P\left(\frac{5}{13}, -\frac{12}{13}\right)$
17. $P\left(-\sqrt{5}/3, \frac{2}{3}\right)$ **19.** $P\left(-\sqrt{2}/3, -\sqrt{7}/3\right)$
21. $t = \pi/4, \left(\sqrt{2}/2, \sqrt{2}/2\right)$; $t = \pi/2, (0, 1)$;
$t = 3\pi/4, \left(-\sqrt{2}/2, \sqrt{2}/2\right)$; $t = \pi, (-1, 0)$;
$t = 5\pi/4, \left(-\sqrt{2}/2, -\sqrt{2}/2\right)$; $t = 3\pi/2, (0, -1)$;
$t = 7\pi/4, \left(\sqrt{2}/2, -\sqrt{2}/2\right)$; $t = 2\pi, (1, 0)$
23. $(1, 0)$ **25.** $(0, -1)$ **27.** $\left(\sqrt{3}/2, -\frac{1}{2}\right)$
29. $\left(-\sqrt{2}/2, -\sqrt{2}/2\right)$ **31.** $\left(-\sqrt{3}/2, \frac{1}{2}\right)$
33. $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ **35.** $\left(-\sqrt{2}/2, -\sqrt{2}/2\right)$
37. (a) $\pi/3$ **(b)** $\pi/3$ **(c)** $\pi/6$ **(d)** $3.5 - \pi \approx 0.36$
39. (a) $2\pi/7$ **(b)** $2\pi/9$ **(c)** $\pi - 3 \approx 0.14$
(d) $2\pi - 5 \approx 1.28$ **41. (a)** $\pi/6$ **(b)** $\left(\sqrt{3}/2, -\frac{1}{2}\right)$
43. (a) $\pi/3$ **(b)** $\left(-\frac{1}{2}, \sqrt{3}/2\right)$
45. (a) $\pi/3$ **(b)** $\left(-\frac{1}{2}, -\sqrt{3}/2\right)$
47. (a) $\pi/4$ **(b)** $\left(-\sqrt{2}/2, -\sqrt{2}/2\right)$
49. (a) $\pi/6$ **(b)** $\left(-\sqrt{3}/2, \frac{1}{2}\right)$
51. (a) $\pi/3$ **(b)** $\left(\frac{1}{2}, \sqrt{3}/2\right)$
53. (a) $\pi/3$ **(b)** $\left(-\frac{1}{2}, -\sqrt{3}/2\right)$
55. $(0.5, 0.8)$ **57.** $(0.5, -0.9)$
59. (a) $\left(-\frac{3}{5}, \frac{4}{5}\right)$ **(b)** $\left(\frac{3}{5}, -\frac{4}{5}\right)$ **(c)** $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ **(d)** $\left(\frac{3}{5}, \frac{4}{5}\right)$

SECTION 5.2 ■ PAGE 416

1. $y, x, y/x$ **2.** 1; 1 **3.** $t = \pi/4$, $\sin t = \sqrt{2}/2$, $\cos t = \sqrt{2}/2$; $t = \pi/2$, $\sin t = 1$, $\cos t = 0$; $t = 3\pi/4$, $\sin t = \sqrt{2}/2$, $\cos t = -\sqrt{2}/2$; $t = \pi$, $\sin t = 0$, $\cos t = -1$; $t = 5\pi/4$, $\sin t = -\sqrt{2}/2$, $\cos t = -\sqrt{2}/2$; $t = 3\pi/2$, $\sin t = -1$, $\cos t = 0$; $t = 7\pi/4$, $\sin t = -\sqrt{2}/2$, $\cos t = \sqrt{2}/2$; $t = 2\pi$, $\sin t = 0$, $\cos t = 1$

5. **(a)** $-\frac{1}{2}$ **(b)** $-\sqrt{3}/2$ **(c)** $\sqrt{3}/3$

7. **(a)** $\sqrt{2}/2$ **(b)** $-\sqrt{2}/2$ **(c)** $-\sqrt{2}/2$

9. **(a)** $-\sqrt{2}/2$ **(b)** $-\sqrt{2}/2$ **(c)** $\sqrt{2}/2$

11. **(a)** $\sqrt{3}/2$ **(b)** $2\sqrt{3}/3$ **(c)** $\sqrt{3}/3$

13. **(a)** $\frac{1}{2}$ **(b)** 2 **(c)** $-\sqrt{3}/2$

15. **(a)** $\sqrt{3}/2$ **(b)** $-2\sqrt{3}/3$ **(c)** $-\sqrt{3}/3$

17. **(a)** -2 **(b)** $2\sqrt{3}/3$ **(c)** $\sqrt{3}$

19. **(a)** $-\sqrt{3}/2$ **(b)** $2\sqrt{3}/3$ **(c)** $-\sqrt{3}/3$

21. **(a)** 0 **(b)** 1 **(c)** 0

23. $\sin 0 = 0$, $\cos 0 = 1$, $\tan 0 = 0$, $\sec 0 = 1$, others undefined

25. $\sin \pi = 0$, $\cos \pi = -1$, $\tan \pi = 0$, $\sec \pi = -1$, others undefined

27. $-\frac{4}{5}, -\frac{3}{5}, \frac{4}{3}$ **29.** $2\sqrt{2}/3, -\frac{1}{3}, -2\sqrt{2}$

31. $\sqrt{13}/7, -6/7, -\sqrt{13}/6$ **33.** $-\frac{12}{13}, -\frac{5}{13}, \frac{12}{5}$ **35.** $\frac{21}{29}, -\frac{20}{29}, -\frac{21}{20}$

37. **(a)** 0.8 **(b)** 0.84147 **39.** **(a)** 0.9 **(b)** 0.93204

41. **(a)** 1 **(b)** 1.02964 **43.** **(a)** -0.6 **(b)** -0.57482

45. Negative **47.** Negative **49.** II **51.** II

53. $\sin t = \sqrt{1 - \cos^2 t}$ **55.** $\tan t = \dfrac{\sin t}{\sqrt{1 - \sin^2 t}}$

57. $\sec t = -\sqrt{1 + \tan^2 t}$ **59.** $\tan t = \sqrt{\sec^2 t - 1}$

61. $\tan^2 t = \dfrac{\sin^2 t}{1 - \sin^2 t}$

63. $\cos t = \frac{3}{5}$, $\tan t = -\frac{4}{3}$, $\csc t = -\frac{5}{4}$, $\sec t = \frac{5}{3}$, $\cot t = -\frac{3}{4}$

65. $\sin t = -2\sqrt{2}/3$, $\cos t = \frac{1}{3}$, $\tan t = -2\sqrt{2}$, $\csc t = -\frac{3}{4}\sqrt{2}$, $\cot t = -\sqrt{2}/4$

67. $\sin t = \frac{12}{13}$, $\cos t = -\frac{5}{13}$, $\csc t = \frac{13}{12}$, $\sec t = -\frac{13}{5}$, $\cot t = -\frac{5}{12}$

69. $\cos t = -\sqrt{15}/4$, $\tan t = \sqrt{15}/15$, $\csc t = -4$, $\sec t = -4\sqrt{15}/15$, $\cot t = \sqrt{15}$

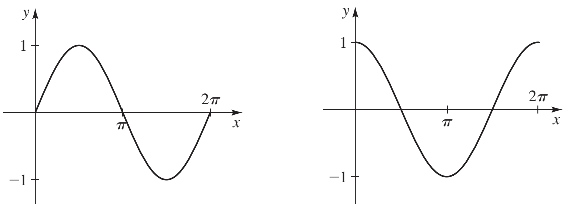
71. Odd **73.** Odd **75.** Even **77.** Neither

79. $y(0) = 4$, $y(0.25) = -2.828$, $y(0.50) = 0$, $y(0.75) = 2.828$, $y(1.00) = -4$, $y(1.25) = 2.828$

81. **(a)** 0.49870 amp **(b)** -0.17117 amp

SECTION 5.3 ■ PAGE 429

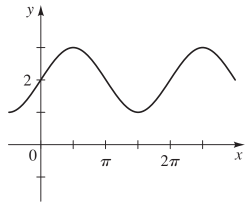
1. $f(t)$; 2π, 1



2. upward; x **3.** $|a|$, $2\pi/k$, 3, π

4. $|a|$, $2\pi/k$, b; 4, $2\pi/3$, $\pi/6$

5.



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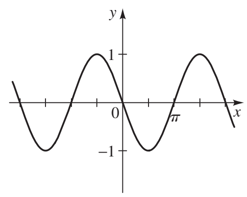

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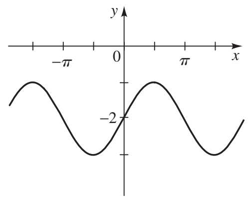

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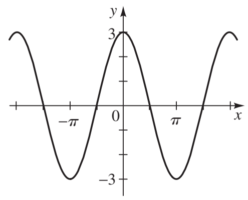

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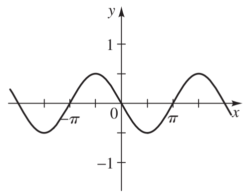

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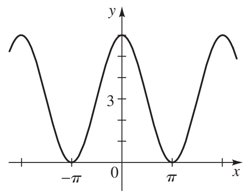
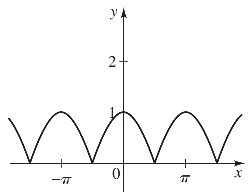
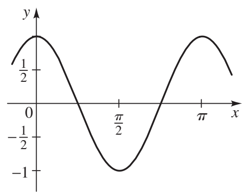


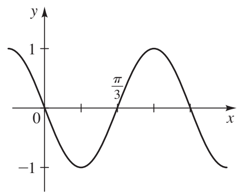
17.



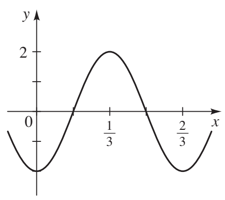
19. 1, π



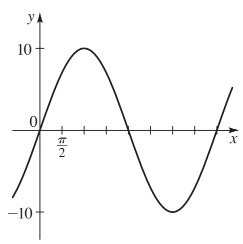
21. 1, $2\pi/3$



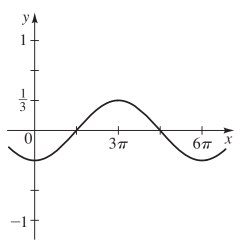
23. 2, $\frac{2}{3}$



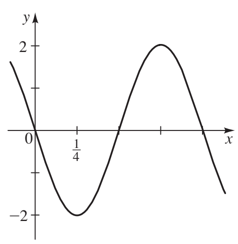
25. 10, 4π



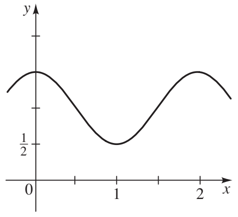
27. $\frac{1}{3}$, 6π



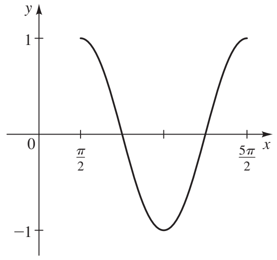
29. 2, 1



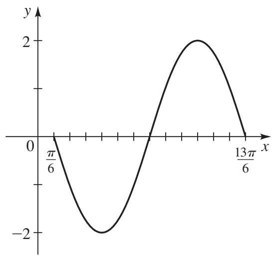
31. $\frac{1}{2}$, 2



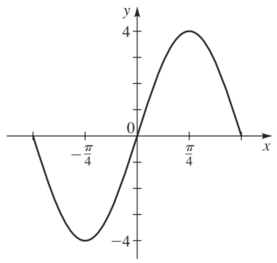
33. 1, 2π, $\pi/2$



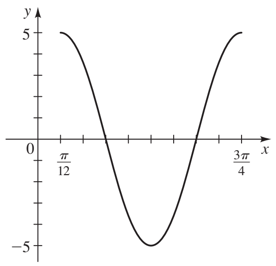
35. 2, 2π, $\pi/6$



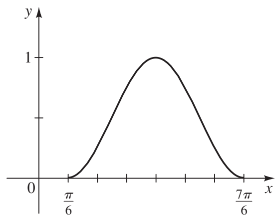
37. 4, π, $-\pi/2$



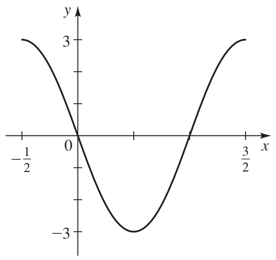
39. 5, $2\pi/3$, $\pi/12$



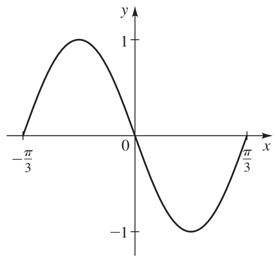
41. $\frac{1}{2}$, π, $\pi/6$



43. 3, 2, $-\frac{1}{2}$



45. 1, $2\pi/3$, $-\pi/3$



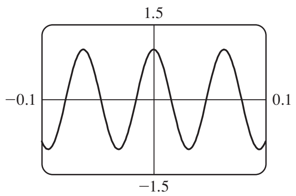
47. **(a)** 4, 2π, 0 **(b)** $y = 4 \sin x$

49. **(a)** $\frac{3}{2}$, $2\pi/3$, 0 **(b)** $y = \frac{3}{2} \cos 3x$

51. **(a)** $\frac{1}{2}$, π, $-\pi/3$ **(b)** $y = -\frac{1}{2} \cos 2(x + \pi/3)$

53. **(a)** 4, $\frac{3}{2}$, $-\frac{1}{2}$ **(b)** $y = 4 \sin 4\pi/3 \left(x + \frac{1}{2}\right)$

55.



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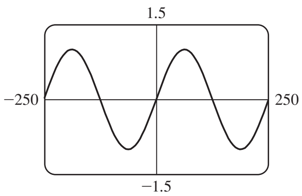

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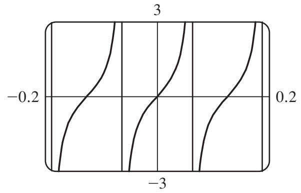

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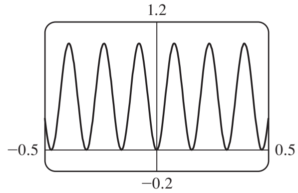

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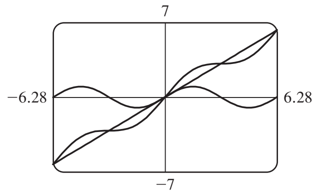

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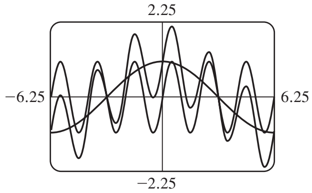

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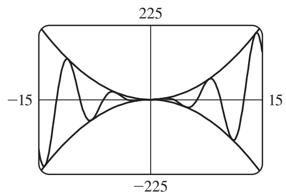
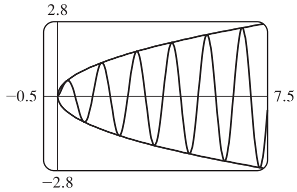


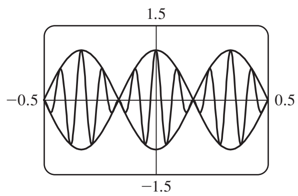
$y = x^2 \sin x$ is a sine curve that lies between the graphs of $y = x^2$ and $y = -x^2$

69.



$y = \sqrt{x} \sin 5\pi x$ is a sine curve that lies between the graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$

71.



$y = \cos 3\pi x \cos 21\pi x$ is a cosine curve that lies between the graphs of $y = \cos 3\pi x$ and $y = -\cos 3\pi x$

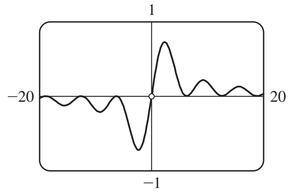
73. Maximum value 1.76 when $x \approx 0.94$, minimum value -1.76 when $x \approx -0.94$ (The same maximum and minimum values occur at infinitely many other values of x.)

75. Maximum value 3.00 when $x \approx 1.57$, minimum value -1.00 when $x \approx -1.57$ (The same maximum and minimum values occur at infinitely many other values of x.)

77. 1.16 **79.** 0.34, 2.80

81. **(a)** Odd **(b)** $\pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$

(c)

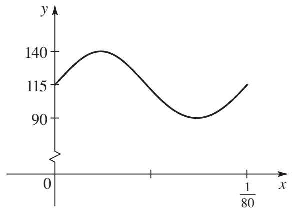


(d) $f(x)$ approaches 0

(e) $f(x)$ approaches 0

83. **(a)** 20 s **(b)** 6 ft

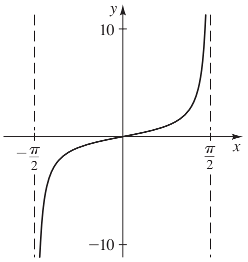
85. (a) $\frac{1}{80}$ min **(b)** 80
(c)



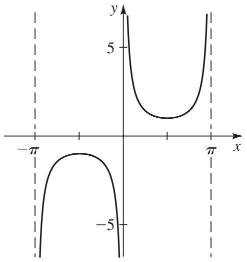
(d) $\frac{140}{90}$; higher than normal

SECTION 5.4 ▪ PAGE 438

1. π; $\frac{\pi}{2} + n\pi$, n an integer

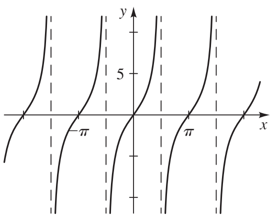


2. 2π; $n\pi$, n an integer

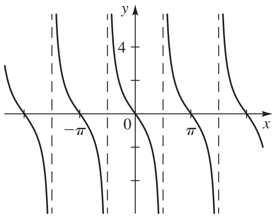


3. II **5.** VI **7.** IV

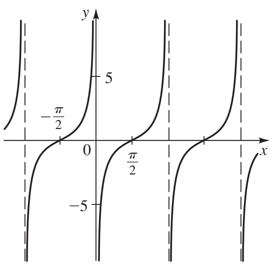
9. π



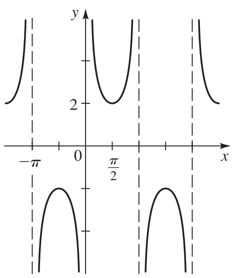
11. π



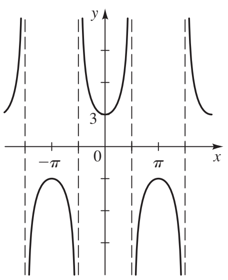
13. π



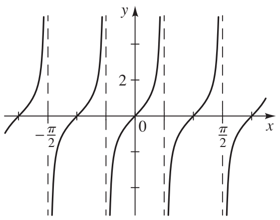
15. 2π



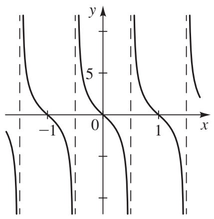
17. 2π



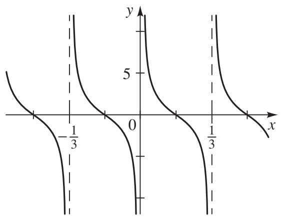
19. $\pi/3$



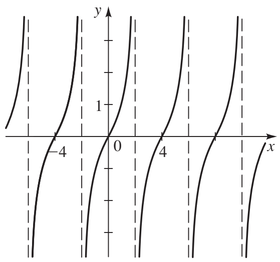
21. 1



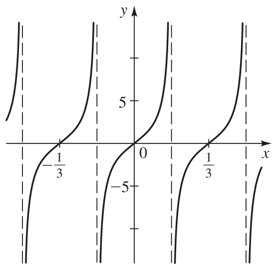
23. $\frac{1}{3}$



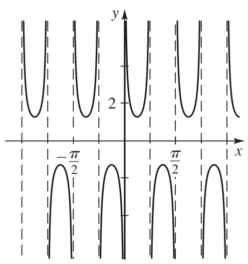
25. 4



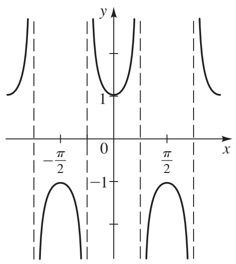
27. $\frac{1}{3}$



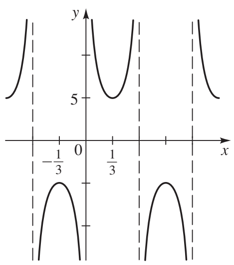
29. $\pi/2$



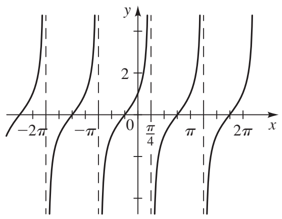
31. π



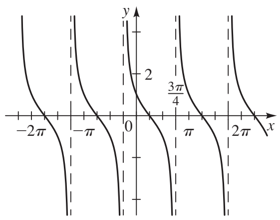
33. $\frac{4}{3}$



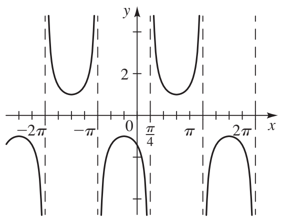
35. π



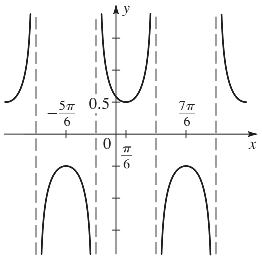
37. π



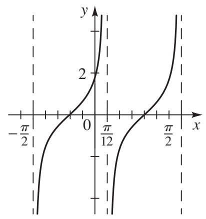
39. 2π



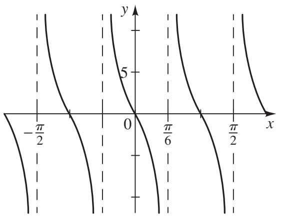
41. 2π



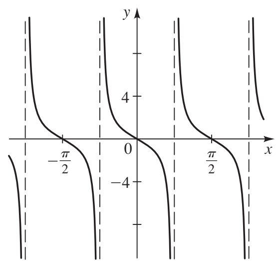
43. $\pi/2$



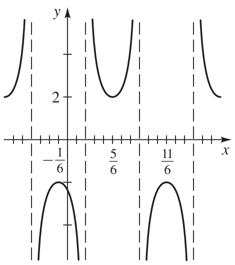
45. $\pi/3$



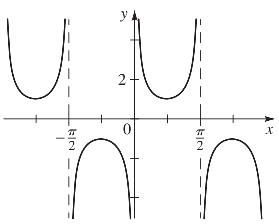
47. $\pi/2$



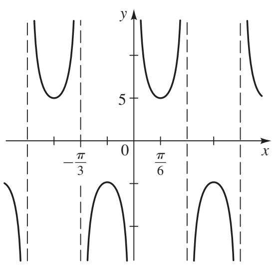
49. 2



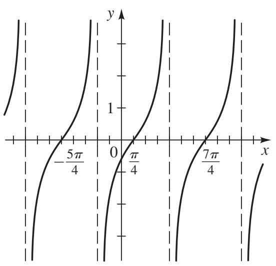
51. π



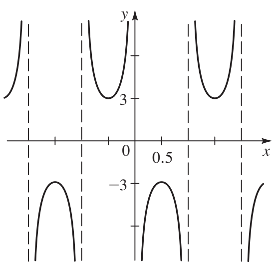
53. $2\pi/3$



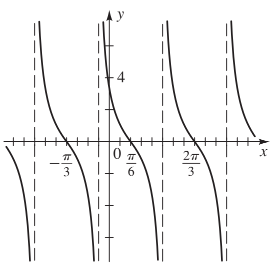
55. $3\pi/2$



57. 2

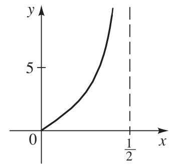


59. $\pi/2$



61. **(a)** 1.53 mi, 3.00 mi, 18.94 mi

(b)



(c) $d(t)$ approaches ∞

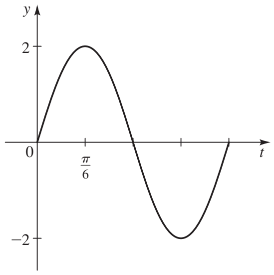
SECTION 5.5 ■ PAGE 444

1. **(a)** $[-\pi/2, \pi/2]$, y, x, $\pi/6$, $\pi/6$, $\frac{1}{2}$ **(b)** $[0, \pi]$; y, x, $\pi/3$, $\pi/3$, $\frac{1}{2}$ **2.** $[-\pi/2, \pi/2]$; (ii) **3.** **(a)** $\pi/2$ **(b)** $\pi/3$ **(c)** Undefined **5.** **(a)** π **(b)** $\pi/3$ **(c)** $5\pi/6$ **7.** **(a)** $-\pi/4$ **(b)** $\pi/3$ **(c)** $\pi/6$ **9.** **(a)** $2\pi/3$ **(b)** $-\pi/4$ **(c)** $\pi/4$ **11.** 0.72973 **13.** 2.01371 **15.** 2.75876 **17.** 1.47113 **19.** 0.88998 **21.** -0.26005 **23.** $\frac{1}{4}$ **25.** 5 **27.** Undefined **29.** $-\frac{1}{5}$ **31.** $\pi/4$ **33.** $\pi/4$ **35.** $5\pi/6$ **37.** $5\pi/6$ **39.** $\pi/4$ **41.** $-\pi/3$ **43.** $\sqrt{3}/3$ **45.** $\frac{1}{2}$ **47.** $-\sqrt{2}/2$

SECTION 5.6 ■ PAGE 456

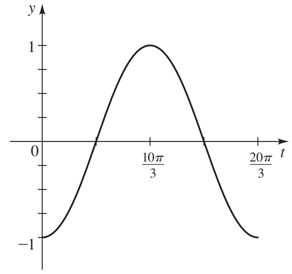
1. **(a)** $a \sin \omega t$ **(b)** $a \cos \omega t$
2. **(a)** $ke^{-ct} \sin \omega t$ **(b)** $ke^{-ct} \cos \omega t$
3. **(a)** $|A|$, $2\pi/k$, b; $A \sin k\left(t - \frac{b}{k}\right)$; b/k **(b)** 5, $\pi/2$, π, $\pi/4$
4. π, $\pi/2$; $\pi/2$, out of phase
5. **(a)** 2, $2\pi/3$, $3/(2\pi)$

(b)

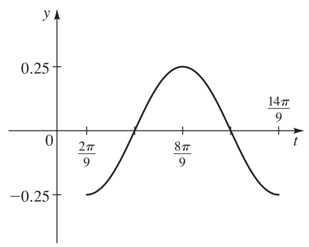


7. **(a)** 1, $20\pi/3$, $3/(20\pi)$

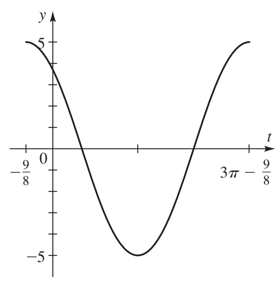
(b)



9. **(a)** $\frac{1}{4}$, $4\pi/3$, $3/(4\pi)$ **(b)**



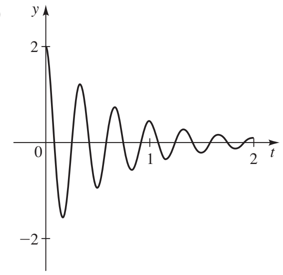
11. **(a)** 5, 3π, $1/(3\pi)$ **(b)**



13. $y = 10\sin\left(\frac{2\pi}{3}t\right)$ **15.** $y = 6\sin(10t)$

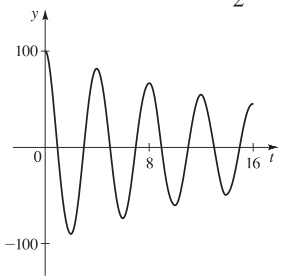
17. $y = 60\cos(4\pi t)$ **19.** $y = 2.4\cos(1500\pi t)$

21. **(a)** $y = 2e^{-1.5t}\cos 6\pi t$ **(b)**

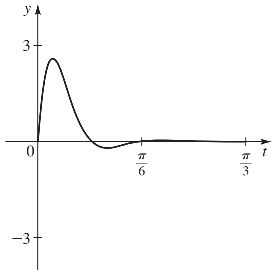


23. **(a)** $y = 100e^{-0.05t}\cos\frac{\pi}{2}t$

(b)

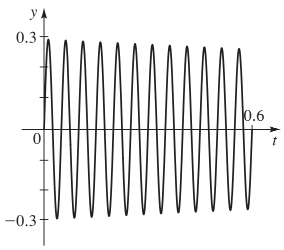


25. **(a)** $y = 7e^{-10t}\sin 12t$ **(b)**



27. **(a)** $y = 0.3e^{-0.2t}\sin(40\pi t)$

(b)



29. $5, \pi, \pi/2, \pi/4$ **31.** $100, 2\pi/5, -\pi, -\pi/5$

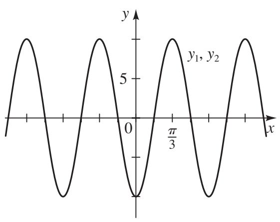
33. $20, \pi, \pi/2, \pi/4$

35. **(a)** $\pi/2, 5\pi/2$

(b) -2π

(c) In phase

(d)

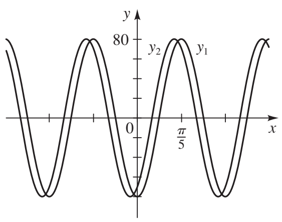


37. **(a)** $\pi/2, \pi/3$

(b) $\pi/6$

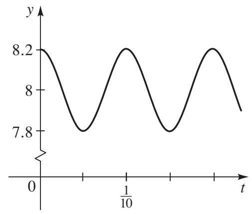
(c) Out of phase

(d)



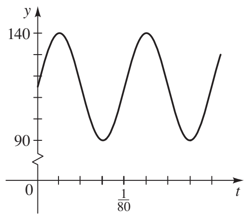
39. **(a)** 10 cycles per minute

(b)



(c) 8.2 m

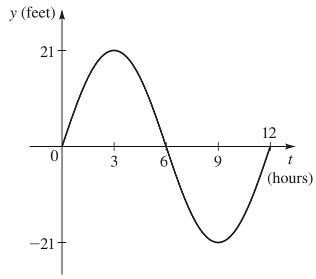
41. **(a)** 25, 0.0125, 80 **(b)**



(c) The period decreases and the frequency increases.

43. $d(t) = 5\sin(5\pi t)$

45. $y = 21\sin\left(\frac{\pi}{6}t\right)$



47. $y = 5\cos(2\pi t)$ **49.** $y = 11 + 10\sin\left(\frac{\pi t}{10}\right)$

51. $y = 3.8 + 0.2\sin\left(\frac{\pi}{5}t\right)$

53. $f(t) = 10\sin\left(\frac{\pi}{12}(t - 8)\right) + 90$

55. **(a)** 45 V **(b)** 40 **(c)** 40 **(d)** $E(t) = 45\cos(80\pi t)$

57. $f(t) = e^{-0.9t}\sin\pi t$ **59.** $c = \frac{1}{3}\ln 4 \approx 0.46$

61. **(a)** $y = \sin(200\pi t), y = \sin\left(200\pi t + \frac{3\pi}{4}\right)$

(b) No; $3\pi/4$

CHAPTER 5 REVIEW ▪ PAGE 463

1. **(b)** $\frac{1}{2}, -\sqrt{3}/2, -\sqrt{3}/3$ **3.** **(a)** $\pi/3$ **(b)** $\left(-\frac{1}{2}, \sqrt{3}/2\right)$

(c) $\sin t = \sqrt{3}/2, \cos t = -\frac{1}{2}, \tan t = -\sqrt{3}, \csc t = 2\sqrt{3}/3,$ $\sec t = -2, \cot t = -\sqrt{3}/3$

5. **(a)** $\pi/4$ **(b)** $(-\sqrt{2}/2, -\sqrt{2}/2)$

(c) $\sin t = -\sqrt{2}/2, \cos t = -\sqrt{2}/2,$ $\tan t = 1, \csc t = -\sqrt{2}, \sec t = -\sqrt{2}, \cot t = 1$

7. **(a)** $\sqrt{2}/2$ **(b)** $-\sqrt{2}/2$ **9.** **(a)** 0.89121 **(b)** 0.45360

11. **(a)** 0 **(b)** Undefined **13.** **(a)** Undefined **(b)** 0

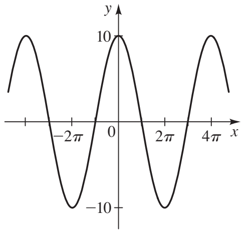
15. **(a)** $-\sqrt{3}/3$ **(b)** $-\sqrt{3}$ **17.** $\frac{\sin t}{1 - \sin^2 t}$ **19.** $\frac{\sin t}{\sqrt{1 - \sin^2 t}}$

21. $\tan t = -\frac{5}{12}, \csc t = \frac{13}{5}, \sec t = -\frac{13}{12}, \cot t = -\frac{12}{5}$

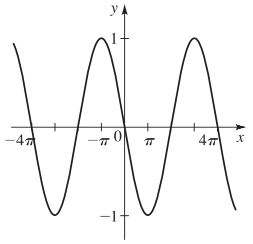
23. $\sin t = 2\sqrt{5}/5$, $\cos t = -\sqrt{5}/5$, $\tan t = -2$, $\sec t = -\sqrt{5}$

25. $-\dfrac{\sqrt{17}}{4} + 4$ **27.** 3

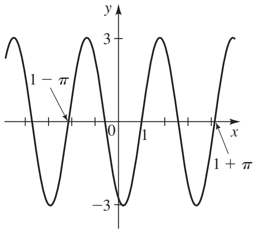
29. **(a)** $10, 4\pi, 0$
(b)



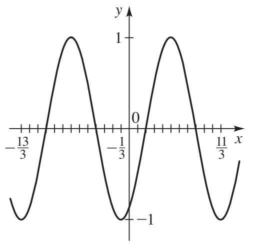
31. **(a)** $1, 4\pi, 0$
(b)



33. **(a)** $3, \pi, 1$
(b)

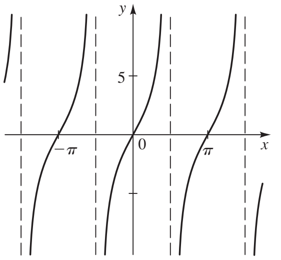


35. **(a)** $1, 4, -\frac{1}{3}$
(b)

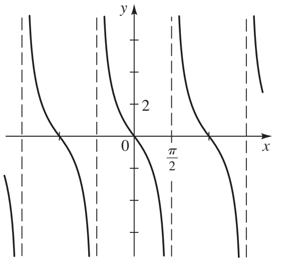


37. $y = 5 \sin 4x$ **39.** $y = \frac{1}{2}\sin 2\pi\left(x + \frac{1}{3}\right)$

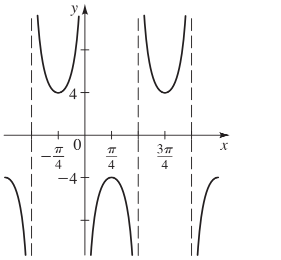
41. π



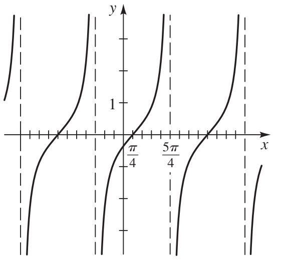
43. π



45. π

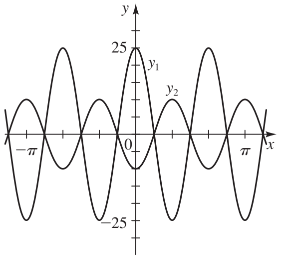


47. 2π

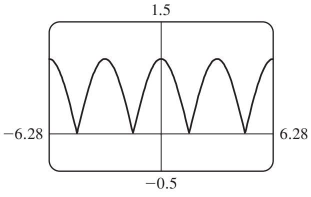


49. $\pi/2$ **51.** $\pi/6$ **53.** $100, \pi/4, -\pi/2, -\pi/16$

55. **(a)** $3\pi/2, 5\pi/2$ **(b)** $-\pi$ **(c)** Out of phase
(d)

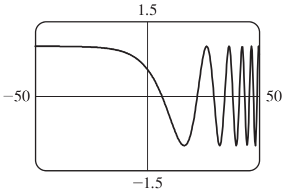


57. **(a)**



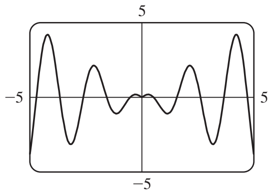
(b) Period π
(c) Even

59. **(a)**



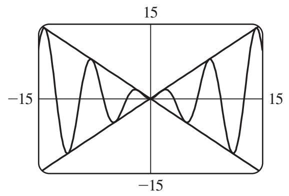
(b) Not periodic
(c) Neither

61. **(a)**



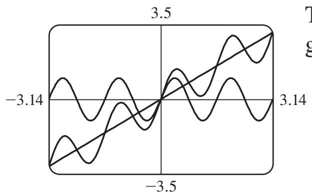
(b) Not periodic
(c) Even

63.



$y = x \sin x$ is a sine function whose graph lies between those of $y = x$ and $y = -x$

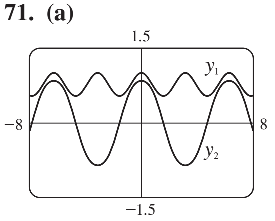
65.



The graphs are related by graphical addition.

67. $1.76, -1.76$ **69.** $0.30, 2.84$

71. **(a)**



(b) y_1 has period π, y_2 has period 2π
(c) $\sin(\cos x) < \cos(\sin x)$, for all x

73. $y = -50 \cos(8\pi t)$

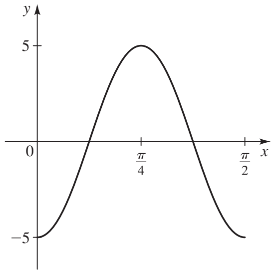
CHAPTER 5 TEST ■ PAGE 465

1. $y = -\frac{5}{6}$ **2.** **(a)** $\frac{4}{5}$ **(b)** $-\frac{3}{5}$ **(c)** $-\frac{4}{3}$ **(d)** $-\frac{5}{3}$
3. **(a)** $-\frac{1}{2}$ **(b)** $-\sqrt{2}/2$ **(c)** $\sqrt{3}$ **(d)** -1

4. $\tan t = -\dfrac{\sin t}{\sqrt{1 - \sin^2 t}}$ **5.** $-\frac{2}{15}$

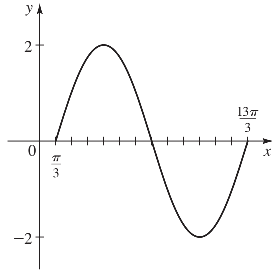
6. (a) $5, \pi/2, 0, 0$

(b)

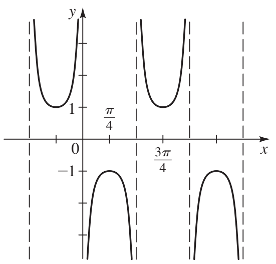


7. (a) $2, 4\pi, \pi/6, \pi/3$

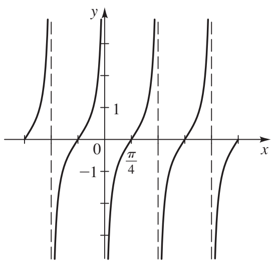
(b)



8. π



9. $\pi/2$



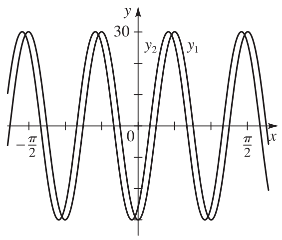
10. (a) $\pi/4$ **(b)** $5\pi/6$ **(c)** 0 **(d)** $\frac{1}{2}$

11. $y = 2 \sin 2(x + \pi/3)$

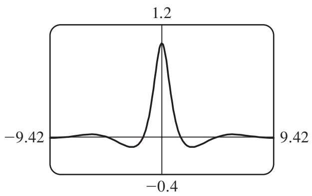
12. (a) $\pi/2, \pi/3$ **(b)** $\pi/6$

(c) Out of phase

(d)



13. (a)

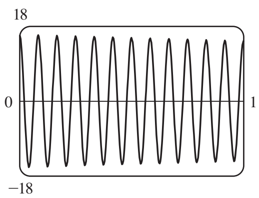


(b) Even

(c) Minimum value -0.11 when $x \approx \pm 2.54$, maximum value 1 when $x = 0$

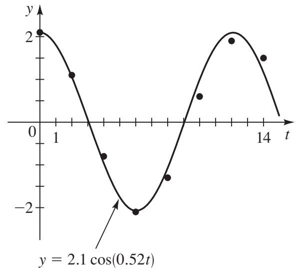
14. $y = 5 \sin(4\pi t)$

15. $y = 16e^{-0.1t} \cos 24\pi t$



FOCUS ON MODELING ■ PAGE 469

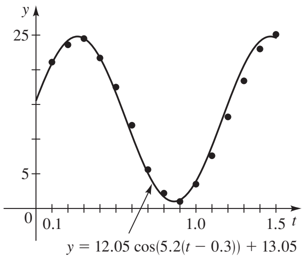
1. (a) and **(c)**



(b) $y = 2.1 \cos(0.52t)$

(d) $y = 2.05 \sin(0.50t + 1.55) - 0.01$ **(e)** The formula of (d) reduces to $y = 2.05 \cos(0.50t - 0.02) - 0.01$. Same as (b), rounded to one decimal.

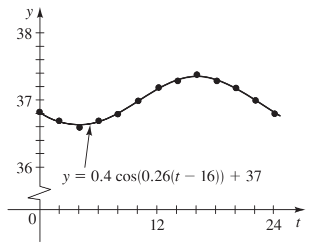
3. (a) and **(c)**



(b) $y = 12.05 \cos(5.2(t - 0.3)) + 13.05$

(d) $y = 11.72 \sin(5.05t + 0.24) + 12.96$ **(e)** The formula of (d) reduces to $y = 11.72 \cos(5.05(t - 0.26)) + 12.96$. Close, but not identical, to (b).

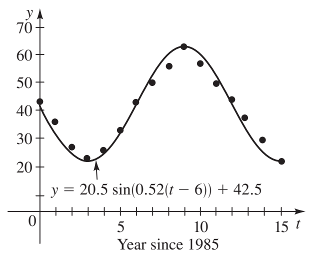
5. (a) and **(c)**



(b) $y = 0.4 \cos(0.26(t - 16)) + 37$, where y is the body temperature (°C) and t is hours since midnight

(d) $y = 0.37 \sin(0.26t - 2.62) + 37.0$

7. (a) and **(c)**



(b) $y = 20.5 \sin(0.52(t - 6)) + 42.5$, where y is the salmon population (× 1000), and t is years since 1985

(d) $y = 17.8 \sin(0.52t + 3.11) + 42.4$