AP Calculus B/C – Compressive Review Quick Notes

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Limits

- **Limit Definition:**

Intuitive description - a limit is an infinite process of approach. For $\lim_{x \to c} f(x)$, we evaluate $f(x)$ at values infinitesimally close to c.

[Informal] $\lim_{x \to c} f(x) = f(c \pm dx)$ **where dx is an infinitesimally x increment.**

Formal Definition - The delta-epsilon definition of a limit has the same meaning but avoids direct reference to infinitesimal values. That means it can work with the real number system, $\mathbb{R}$ (neither dx nor ∞ exist in $\mathbb{R}$). It works because an interval contains an infinite number of discrete values. [Larson p. 52]

[Formal] $\lim_{x \to c} f(x) = L$ **if and only if**

There exists a $\delta > 0$ for each $\varepsilon > 0$, such that $0 < |x - c| < \delta$ and $|f(x) - L| < \varepsilon$.

Note: $\delta \to$ interval of x $\quad$ $\varepsilon \to$ interval of $f(x)$

- **Single sided limits**

Left hand Limit	$\lim_{x \to c^-} f(x)$	Approach from Below ($x < c$), x **increases toward c**.	$\lim_{x \to \infty} f(x)$ is a form of left hand limit
Right hand Limit	$\lim_{x \to c^+} f(x)$	Approach from Above ($x > c$), x **decreases toward c**.	$\lim_{x \to -\infty} f(x)$ is a form of right hand limit

- **Asymptotes**

Vertical	$\lim_{x \to c^-} f(x) = \pm\infty$	Or	$\lim_{x \to c^+} f(x) = \pm\infty$	(where c is finite)
Horizontal	$\lim_{x \to \infty} f(x) = L$	Or	$\lim_{x \to -\infty} f(x) = L$	(where L is finite)

The above excludes slant asymptotes which occur only in rational functions with numerator 1 degree higher than denominator

- **Continuity**

Definition - $\lim_{x \to c} f(x) = f(c)$

Point Continuity: The function value at a location equals its limit at that location.

Interval Continuity: Above must be true at every point in the interval. Continuity at interval endpoints is defined by single sided limits which stay within the interval.

This implies that the limit and function value actually exist: if either are undefined, the function is discontinuous by definition.

- **Nonexistent limits** If any one of the below are true it means that a limit does not exist.

Left Limit ≠ Right Limit	$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$	Example:	$\lim_{x \to 0} \frac{\lvert x \rvert}{x}$ **Does Not Exist**
Infinite or Divergent Limits	$\lim_{x \to c} f(x) = \pm\infty$	Example:	$\lim_{x \to 2} \frac{1}{(x-2)^2}$ **Does Not Exist**
Function Changes Values Infinitely Rapid Near Limit	**Note: Limit may exist if oscillations → 0 amplitude at c*	Example:	$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ **Does Not Exist**

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- **Discontinuities**

Removable $\lim_{x\to c^-} f(x)$ Exists, $f(c)$ Does Not Exist

Example: $\lim_{x\to 0}\frac{\sin x}{x}=1$

Removable discontinuities always occur where the numerator and denominator both have zeros.

Non-Removable $\lim_{x\to c} f(x)$ Does Not Exist [See Nonexistent Limits for examples]

It does not matter if $f(c)$ is explicitly defined, the function has a discontinuity at $x=c$ if the limit does not exist for any reason.

- **Evaluating limits**

Evaluate at limit point

Anywhere a function is continuous it can simply be evaluated

$$\lim_{x\to c} f(x)=f(c)$$

Example: $\lim_{x\to 0}\frac{e^x}{x+1}=\frac{e^0}{0+1}=1$

Algebraically Cancel

When removable discontinuities are present, there may be a common factor in numerator and denominator

Example: $\lim_{x\to 1}\frac{x+1}{x^2+3x+2}\ \left(\to\frac{0}{0}\right)$

$$\lim_{x\to 1}\frac{x+1}{x^2+3x+2}=\lim_{x\to 1}\frac{x+1}{(x+2)(x+1)}=\lim_{x\to 1}\frac{1}{x+2}=\frac{1}{3}$$

Numeric Method

Technically successive approximation. The accuracy of a numeric method will depend on the function and the step size.

Example: $\lim_{x\to 0}\frac{\sin(x)}{x}\ \left(\to\frac{0}{0}\right)$

x	-0.25	-0.1	-0.01	0	0.01	0.1	0.25
$f(x)$	**0.989616**	**0.998334**	**0.999983**	Infer from Pattern	**0.999983**	**0.998334**	**0.989616**

It is often helpful to decrease the order of magnitude between steps as the limit value is approached.
The numeric method is exactly equal to the limit if x could be infinitely close on both sides.

Squeeze Theorem

If $g(x)\ge f(x)\ge h(x)$ in the neighborhood of c,
And $\lim_{x\to c} g(x)=a=\lim_{x\to c} h(x)$,
then $\lim_{x\to c} f(x)=a$

Example: $\lim_{x\to 0}\frac{\sin(x)}{x}\ \left(\to\frac{0}{0}\right)$

$$-x^2+1\le\frac{\sin(x)}{x}\le x^2+1\quad\to\quad\lim_{x\to 0}-x^2+1=1\ ;\ \lim_{x\to 0}-x^2+1=1\quad\to\quad\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

L'Hopital's Rule

If $\lim_{x\to c}\frac{f(x)}{g(x)}\to\frac{0}{0}$ or $\frac{\infty}{\infty}$,

Then $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$

Example: $\lim_{x\to 0}\frac{\sin(x)}{x}\ \left(\to\frac{0}{0}\right)$

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- **Resolving Various Indeterminate Limit Forms with L'Hopital's Rule**

Conversion of limit type - All indeterminate limit forms can be algebraically converted to $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Conversion to either type allows for subsequent application of L'Hopital's Rule to either find the limit or confirm its nonexistence.

Limit type		Conversion Process	
$0 \cdot \infty$	$L = \lim_{x\to c} f(x)\cdot g(x)$ $\left(\lim_{x\to c} f(x) = 0\,;\ \lim_{x\to c} g(x) = \infty\right)$	Substitute $g(x) = \frac{1}{(g(x))^{-1}}$	$L = \lim_{x\to c} \frac{f(x)}{(g(x))^{-1}} \to \frac{0}{0}$
1^{∞}	$L = \lim_{x\to c} f(x)^{g(x)}$ $\left(\lim_{x\to c} f(x) = 1\,;\ \lim_{x\to c} g(x) = \infty\right)$	Apply $\ln(\ )$ to both sides, then Substitute $g(x) = \frac{1}{(g(x))^{-1}}$	$\ln(L) = \lim_{x\to c} \frac{\ln(f(x))}{(g(x))^{-1}} \to \frac{0}{0}$
0^{0}	$L = \lim_{x\to c} f(x)^{g(x)}$ $\left(\lim_{x\to c} f(x) = 0\,;\ \lim_{x\to c} g(x) = 0\right)$	Apply $\ln(\ )$ to both sides, then Substitute $g(x) = \frac{1}{(g(x))^{-1}}$	$\ln(L) = \lim_{x\to c} \frac{\ln(f(x))}{(g(x))^{-1}} \to -\frac{\infty}{\infty}$
∞^{0}	$L = \lim_{x\to c} f(x)^{g(x)}$ $\left(\lim_{x\to c} f(x) = \infty\,;\ \lim_{x\to c} g(x) = 0\right)$	Apply $\ln(\ )$ to both sides, then Substitute $g(x) = \frac{1}{(g(x))^{-1}}$	$\ln(L) = \lim_{x\to c} \frac{\ln(f(x))}{(g(x))^{-1}} \to \frac{\infty}{\infty}$
$\infty - \infty$	$L = \lim_{x\to c} f(x) - g(x)$ $\left(\lim_{x\to c} f(x) = \infty\,;\ \lim_{x\to c} g(x) = \infty\right)$	Substitute $g(x) = \frac{1}{(g(x))^{-1}}$ then Substitute $g(x) = \frac{1}{(g(x))^{-1}}$ then Combine terms with common denominator	$L = \lim_{x\to c} \frac{f(x)}{(g(x))^{-1}}$

- **Determinate Limit Forms (possibly mistaken for indeterminate)**

$\infty \cdot \infty \quad = \infty$	$\lim_{x\to c} f(x)\cdot g(x) = \infty$	$\left(\lim_{x\to c} f(x) = \infty\,;\ \lim_{x\to c} g(x) = \infty\right)$
$0^{\infty} \quad = 0$	$\lim_{x\to c} f(x)^{g(x)} = 0$	$\left(\lim_{x\to c} f(x) = 0\,;\ \lim_{x\to c} g(x) = \infty\right)$
$0^{-\infty} \quad = \infty$	$\lim_{x\to c} f(x)^{g(x)} = \infty$	$\left(\lim_{x\to c} f(x) = 0;\ \lim_{x\to c} g(x) = -\infty\right)$
$0^{\infty} \quad = 0$	$\lim_{x\to c} f(x)^{g(x)} = 0$	$\left(\lim_{x\to c} f(x) = 0\,;\ \lim_{x\to c} g(x) = \infty\right)$
$\infty^{-\infty} \quad = 0$	$\lim_{x\to c} f(x)^{g(x)} = \infty$	$\left(\lim_{x\to c} f(x) = \infty\,;\ \lim_{x\to c} g(x) = -\infty\right)$
$\frac{0}{\infty} \quad = 0$	$\lim_{x\to c} \frac{f(x)}{g(x)} = 0$	$\left(\lim_{x\to c} f(x) = 0\,;\ \lim_{x\to c} g(x) = \infty\right)$
$\frac{\infty}{0} \quad = \infty$	$\lim_{x\to c} \frac{f(x)}{g(x)} = 0$	$\left(\lim_{x\to c} f(x) = \infty\,;\ \lim_{x\to c} g(x) = 0\right)$