8.4.32 $\int \frac{1}{x\sqrt{4x^2+16}}\,dx$

$4x^2+16 = 4(x^2+4)$

"$\tan^2\theta + 1 = \sec^2\theta$"

$x = 2\tan\theta$

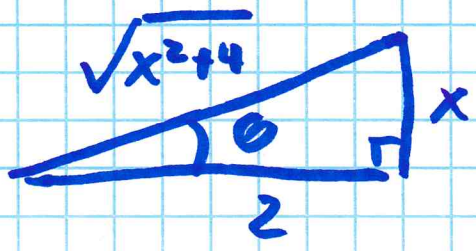
$4(x^2+4) = 4(4\tan^2\theta+4)$

$16(\tan^2\theta+1) = 16\sec^2\theta$

$dx/d\theta = 2\sec^2\theta$

$dx = 2\sec^2\theta\,d\theta$

$\tan\theta = \frac{x}{2}$



$\cot\theta = \frac{2}{x}$

$\csc\theta = \sqrt{x^2+4}\cdot\frac{1}{x}$

$= \int \frac{2\sec^2\theta\,d\theta}{2\tan\theta\,4\sec\theta}$

$= \frac{1}{4}\int \sec^{-1}\theta\tan^{-1}\theta\,d\theta$

$= \frac{1}{4}\int \frac{1}{\cos\theta}\,\frac{\cos\theta}{\sin\theta}\,d\theta$

$= \frac{1}{4}\int \csc\theta\,d\theta$

$= \frac{-1}{4}\ln|\csc\theta+\cot\theta| + C$

$= \frac{-1}{4}\ln\left|\frac{1}{x}\left(2+\sqrt{x^2+4}\right)\right| + C$

~~$\frac{1}{4}\ln|x| - \frac{1}{8}\ln|x^2+4|$~~

$= \frac{1}{4}\ln|x| - \frac{1}{4}\ln|2+\sqrt{x^2+4}| + C$

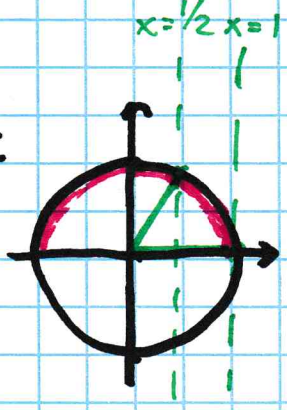
8.4.52 $\int_3^6 x^{-2}\sqrt{x^2-9}\,dx = I$ "$\sec^2\theta - 1 = \tan^2\theta$"

$x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$

$x/3 = \sec\theta$, $\frac{3}{x} = \cos\theta$, $\theta = \arccos 3/x$

$\theta_2 = \arccos 1/2$, $\theta_1 = \arccos 1$

$\theta_2 = \pi/3$ $\theta_1 = 0$



$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$

$$I = \int_0^{\pi/3} (3\sec\theta)^{-2}\, 3\tan\theta\, 3\sec\theta\tan\theta\,d\theta$$

$$= \int_0^{\pi/3} \sec^{-1}\theta\,\tan^2\theta\,d\theta = \int_0^{\pi/3} \sec^{-1}\theta\,\sec^2\theta - 1\,d\theta$$

$$= \int_0^{\pi/3} \sec\theta\,d\theta - \int_0^{\pi/3} \cos\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta\Big|_0^{\pi/3}$$

$$= \ln|2+\sqrt{3}| - \sqrt{3}/2 - \ln|1+0|.$$

8.4.46

$$\int \frac{x}{\sqrt{x^2-6x+5}}\,dx = I$$

$$x^2-6x+5 = x^2-6x+9-9+5 = (x-3)^2-4$$

Let $x-3 = 2\cdot\sec\theta$, $\quad x = 2\sec\theta+3$

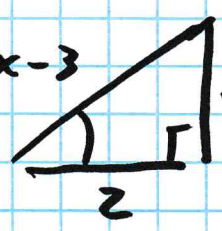
$$x^2-6x+5 = 4\sec^2\theta-4 = 4\tan^2\theta$$

$$\frac{d}{d\theta}(x-3) = \frac{d}{d\theta}(2\sec\theta) \Rightarrow dx = 2\sec\theta\tan\theta\,d\theta$$

$$I = \int \frac{2\sec\theta+3}{2\tan\theta}\,2\sec\theta\tan\theta\,d\theta$$

$$= 2\int \sec^2\theta\,d\theta + 3\int \sec\theta\,d\theta$$

$$= 2\tan\theta + 3\ln|\sec\theta+\tan\theta| + C$$



$$\sec\theta = \frac{x-3}{2} \qquad \tan\theta = \frac{1}{2}\sqrt{x^2-6x+5}$$

$$I = \sqrt{x^2-6x+5} + 3\ln\left|\frac{x-3}{2} + \frac{\sqrt{x^2-6x+5}}{2}\right| + C$$