Def. of a Series

A series is the sum of a sequence.

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + \overbrace{a_2}^{\text{single term}} + a_3 + \cdots}_{\text{series a}}$$

generating function for nth term

Def. of a Series Sum

The sum of a series is limit of its sequence of Partial Sums.

Let $\underbrace{S_k}_{\text{sequence of Partial Sums.}} = \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_{k-1} + a_k$

$S_2 = \sum_{n=1}^{2} a_n = a_1 + a_2$

$S_3 = \sum_{n=1}^{3} a_n = a_1 + a_2 + a_3$

$\vdots$

Convergence of a Series - The sum of the series exists

Divergence of a Series - The Series Sum DNE for <u>Any</u> <u>Reason</u>

Two major issues for convergence

1. tendency to ∞ or $-\infty$
2. oscillations that do not shrink

Ex. $\displaystyle\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 \ldots$

$S_1 = 1$ $\quad\quad \displaystyle\lim_{n\to\infty} S_k = DNE$

$S_2 = 0$

$S_3 = 1$

$S_4 = 0$

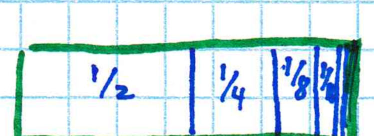
$\vdots$

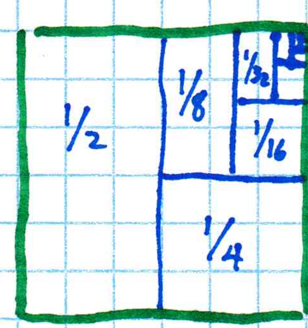
Ex. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$

$$\left.\begin{aligned} S_1 &= \tfrac{1}{2} \\ S_2 &= \tfrac{3}{4} \\ S_3 &= \tfrac{7}{8} \\ S_4 &= \tfrac{15}{16} \end{aligned}\right\} S_k = \frac{2^k - 1}{2^k}$$

$\vdots$

$\displaystyle\lim_{k\to\infty} S_k = 1$





⚠ determining series convergence by definition is unreliable because the generating function for partial sum may be unclear

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \cdots$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Basel Problem

S_k

$S_1 = 1$

$S_2 = \frac{5}{4}$

$S_3 = \frac{52}{36}$

$S_4 = \frac{205}{144}$

$S_5 = \frac{5269}{3600}$

$\vdots$

$S_k \overset{?}{=} f(k)$

⚠ Series Pitfalls

The associative & commutative properties Do Not hold for all series.

(commutation & association require that a series is Absolutely Conv.)

Commutative Prop. of Addition

$$a + b = b + a \quad \text{(reordering)}$$

Associative Prop. of Addition

$$a + (b + c) = (a + b) + c = a + b + c$$

Ex. Let $S = 1 - 1 + 1 - 1 \ldots$

⚠ uses comm. & Ass.

$S = \lim S_k$.

Assoc. 1

Let $S = (1-1) + (1-1) + (1-1) \cdots = 0$

$S_k = 0$ $\quad 0 \quad 0 \quad 0 \cdots S = 0$?

Assoc. 2

Let $S = 1 + (-1+1) + (-1+1) + (-1+1) \cdots$

$S_k = \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$

$S = 1$?

$1 = 0$ ← X