Convergence Tests & Theorems: 9.2

nth Term Divergence Test

$\sum_{n=1}^{\infty} a_n$ Diverges if $\lim_{n\to\infty} a_n \neq 0$.

[If the series terms are not shrinking, the series cannot have a finite sum.]

Geometric Series

A series is geometric if it has a form

$$\sum a \cdot r^n$$

exponential dependence on index

common ratio

$$\sum_{n=0}^{\infty} a \cdot r^n = \frac{a}{1-r} \text{ if } |r| < 1, \text{ Diverges if } |r| \geq 1$$

$$\sum_{n=1}^{\infty} a \cdot r^n = \frac{ar}{1-r} \text{ if } |r| < 1, \text{ Diverges if } |r| \geq 1$$

By Definition Convergence

Let $S_k = \sum_{n=1}^{k} a_n$. if $\lim_{n\to\infty} S_k$ exists $\sum a$ converges.

[Use on <u>telescoping</u> series or as last resort.]

Def. of a Telescoping Series

A series is Telescoping if it has form

$$\sum_{n} a_n = \sum_{n} (b_n - b_{n+1})$$

Where

$$a_n = (b_n - b_{n+1}) \; ; \quad \begin{aligned} a_1 &= b_1 - b_2 \\ a_2 &= b_2 - b_3 \\ &\vdots \end{aligned}$$

Ex. Geom. Series

- $\sum a_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots$

  $\sum a_n$ ~~ge~~ is geometric with $r = \frac{1}{2}$, $\sum a = \frac{(1)\frac{1}{2}}{1-\frac{1}{2}} = 1$

- $\sum a = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = .\overline{3}$

  $\sum_{n=1}^{\infty} 3 \cdot \frac{1}{10}^n = \frac{3\frac{1}{10}}{1-\frac{1}{10}} = \frac{3/10}{9/10} = \frac{1}{3}$

  or

  $\sum_{n=0}^{\infty} \frac{3}{10} \frac{1}{10}^n$

- $\sum a_n = .\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} \cdots = \sum_{n=1}^{\infty} 9 \frac{1}{10}^n = \frac{9\frac{1}{10}}{1-\frac{1}{10}}$

  $.\overline{9} = 1$

- $\sum a_n = .157\overline{57} = .1 + .0\overline{57} = \frac{1}{10} + \frac{57}{1000} + \frac{57}{10000} \cdots$

  $= \frac{1}{10} + \sum_{n=0}^{\infty} \frac{57}{1000} \cdot \left(\frac{1}{100}\right)^n = \frac{57}{1000} \cdot \frac{1}{1-\frac{1}{100}} = \frac{57}{1000} \cdot \frac{100}{99} + \frac{1}{10}$

  $= \frac{1}{10} + \frac{57}{990} = \frac{99}{990} + \frac{57}{990} = \frac{156}{990}$

9.2.57 Let $\sum a_n = \sum_{n=1}^{\infty} \frac{n+10}{10n+1}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+10}{10n+1} = \frac{1}{10} \neq 0$. $\sum a$ diverges by nth Term test.

9.2.59 Let $\sum a_n = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

note: $\sum a_n$ is telescoping: $\sum_n a_n = \sum_n (b_n - b_{n+2})$ with $b_n = \frac{1}{n}$

Let $S_k = \sum_{n=1}^{k} a_n$. $S_1 = 1 - \frac{1}{3}$, $S_2 = (1-\frac{1}{3}) + (\frac{1}{2} - \frac{1}{4})$
$= 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$

$S_3 = (\frac{1}{1} - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$

$S_4 = (1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{4} - \frac{1}{6})$
$= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$

$S_k = 1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}$ for $k \geq 2$

$\lim_{k\to\infty} S_k = \frac{3}{2}$. $\sum a_n = \frac{3}{2}$