Improper Integrals

I. Horizontal Asymptote (Infinite bound of Integration)

Improper

$$\int_a^{\infty} f(x)\,dx \quad ; \qquad \int_a^b f(x)\,dx = F(b) - F(a)$$

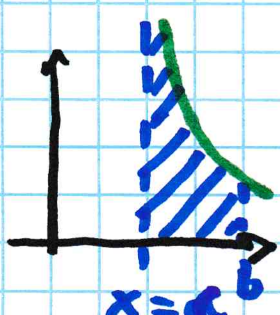
FToC

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$= \lim_{b\to\infty} F(b) - F(a)$$

Improper integral Converges if Limit exists. It Diverges if Lim DNE for any reason.

II. Vert. Asymptote



$$\int_c^b f(x)\,dx = F(b) - \lim_{a\to c^+} F(a)$$

Improper

8.8.23

$$\int_0^\infty e^{-x} \cos x \, dx$$

$$I = \int e^{-x} \cos x \, dx$$

$u = \cos x \qquad du = -\sin x \, dx$

$dv = e^{-x} dx \qquad v = -e^{-x}$

$$= -e^{-x} \cos x - \int e^{-x} \sin x \, dx$$

$u_2 = \sin x \qquad du_2 = \cos x \, dx$

$dv = e^{-x} dx \qquad v = -e^{-x}$

$$= -e^{-x} \cos x + e^{-x} \sin x - \underbrace{\int e^{-x} \cos x \, dx}_{I}$$

$$I = \int e^{-x} \cos x \, dx = \frac{1}{2} e^{-x} (\sin x - \cos x) + C$$

$$\int_0^\infty e^{-x} \cos x \, dx = \underbrace{\lim_{b \to \infty} I(b)}_{0} - \underbrace{I(0)}_{-1/2} = 1/2$$

Converges

$$\underbrace{\lim_{b \to \infty} \frac{\sin b - \cos b}{2e^{b}} = 0}$$

$$= \frac{1}{2} \lim_{b \to \infty} \frac{\sin b}{e^{b}} - \frac{1}{2} \lim_{b \to \infty} \frac{\cos b}{e^{b}}$$

8.8.41 $\int_{2}^{4} \frac{2}{x\sqrt{x^2-4}}\,dx$

V.A. $\rightarrow$ 2

$I = 2\int x^{-1}(x^2-4)^{-1/2}\,dx$ Let $x = 2\sec\theta$

$x^2 - 4 = 4\sec^2\theta - 4 = 4\tan^2\theta$

$(x^2-4)^{-1/2} = (2\tan\theta)^{-1}$; $dx = 2\sec\theta\tan\theta\,d\theta$

$I = 2\int \frac{2\sec\theta\tan\theta}{2\sec\theta\; 2\tan\theta}\,d\theta = \int d\theta = \theta + C$

$\frac{x}{2} = \sec\theta \Rightarrow \frac{2}{x} = \cos\theta$

$\theta = \arccos\left(\frac{2}{x}\right)$

arccos $\frac{1}{2}$

$\int_2^4 2x^{-1}(x^2-4)^{-1/2}\,dx = I(4) - \underbrace{\lim_{a\to 2^+} I(a)}_{0}$

$= \pi/3$, converges

$\lim_{a\to 2^+} \arccos\left(\frac{2}{a}\right) = 0$