AP® CALCULUS BC
2010 SCORING GUIDELINES

Question 5

Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

(a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

(a) $$f\left(\frac{1}{2}\right) \approx f(1) + \left(\left.\frac{dy}{dx}\right|_{(1,\,0)}\right) \cdot \Delta x$$

$$= 0 + 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

$$f(0) \approx f\left(\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(\frac{1}{2},\, -\frac{1}{2}\right)}\right) \cdot \Delta x$$

$$\approx -\frac{1}{2} + \frac{3}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{5}{4}$$

2 : { 1 : Euler's method with two steps
1 : answer }

(b) Since f is differentiable at $x = 1$, f is continuous at $x = 1$. So, $\lim_{x \to 1} f(x) = 0 = \lim_{x \to 1} \left(x^3 - 1\right)$ and we may apply L'Hospital's Rule.

$$\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{\lim_{x \to 1} f'(x)}{\lim_{x \to 1} 3x^2} = \frac{1}{3}$$

2 : { 1 : use of L'Hospital's Rule
1 : answer }

(c) $$\frac{dy}{dx} = 1 - y$$

$$\int \frac{1}{1 - y}\, dy = \int 1\, dx$$

$$-\ln|1 - y| = x + C$$

$$-\ln 1 = 1 + C \Rightarrow C = -1$$

$$\ln|1 - y| = 1 - x$$

$$|1 - y| = e^{1-x}$$

$$f(x) = 1 - e^{1-x}$$

5 : { 1 : separation of variables
1 : antiderivatives
1 : constant of integration
1 : uses initial condition
1 : solves for y }

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

Solution	Scoring
(a) $\left.\frac{dW}{dt}\right\rvert_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$ The tangent line is $y = 1400 + 44t$. $W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons	$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$
(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \ge 1400$ Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \le t \le \frac{1}{4}$. The answer in part (a) is an underestimate.	$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$
(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ $\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$ $\ln\lvert W - 300\rvert = \frac{1}{25}t + C$ $\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$ $W - 300 = 1100e^{\frac{1}{25}t}$ $W(t) = 300 + 1100e^{\frac{1}{25}t}, \ 0 \le t \le 20$	$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables

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Question 5

Consider the differential equation $\frac{dy}{dx} = y^2(2x + 2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x \to 0} \frac{f(x) + 1}{\sin x}$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) $\lim_{x \to 0}(f(x) + 1) = -1 + 1 = 0$ and $\lim_{x \to 0} \sin x = 0$

Using L'Hospital's Rule,

$$\lim_{x \to 0} \frac{f(x) + 1}{\sin x} = \lim_{x \to 0} \frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos 0} = \frac{(-1)^2 \cdot 2}{1} = 2$$

2 : { 1 : L'Hospital's Rule
1 : answer }

(b) $$f\left(\frac{1}{4}\right) \approx f(0) + f'(0)\left(\frac{1}{4}\right) = -1 + (2)\left(\frac{1}{4}\right) = -\frac{1}{2}$$

$$f\left(\frac{1}{2}\right) \approx f\left(\frac{1}{4}\right) + f'\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = -\frac{1}{2} + \left(-\frac{1}{2}\right)^2\left(2 \cdot \frac{1}{4} + 2\right)\left(\frac{1}{4}\right) = -\frac{11}{32}$$

2 : { 1 : Euler's method
1 : answer }

(c) $$\frac{dy}{dx} = y^2(2x + 2)$$

$$\frac{dy}{y^2} = (2x + 2)\,dx$$

$$\int \frac{dy}{y^2} = \int (2x + 2)\,dx$$

$$-\frac{1}{y} = x^2 + 2x + C$$

$$-\frac{1}{-1} = 0^2 + 2 \cdot 0 + C \Rightarrow C = 1$$

$$-\frac{1}{y} = x^2 + 2x + 1$$

$$y = -\frac{1}{x^2 + 2x + 1} = -\frac{1}{(x + 1)^2}$$

5 : { 1 : separation of variables
1 : antiderivatives
1 : constant of integration
1 : uses initial condition
1 : solves for y }

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

MC.1	D
MC.2	B
MC.3	C
MC.4	D
MC.5	C
MC.6	D
MC.7	C
MC.8	C
MC.9	B
MC.10	D